

***h*-STABILITY ANALYSIS AND CONTROLLER DESIGN FOR SINGULAR SYSTEMS**

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Abstract: *This paper proposes novel sufficient conditions for establishing h -stability and synthesizing controllers for linear continuous-time singular systems. The analysis is carried out through the Lyapunov Krasovskii functional method, offering a generalized perspective of exponential stability. To demonstrate the effectiveness of the proposed approach, several numerical examples are presented.*

Keywords: *Exponential stability, h -stability, Lyapunov-Krasovskii functional, controller design, singular systems, linear matrix inequality.*

1. Introduction

The concept of h -stability, originally introduced by Pinto in [1,2], represents a stronger form of stability than traditional exponential stability and has proven particularly valuable in the analysis of differential and difference systems subject to specific perturbations. Medina and Pinto further broadened this notion by introducing the class of h -systems in [3], extending asymptotic stability analysis to cover a wider range of system behaviors. Unlike classical exponential stability, analyzing stability under the h -stability framework introduces additional complexities, yet offers greater flexibility and applicability, especially in systems exhibiting non-exponential convergence. This has led to a growing body of work on the topic [4,5], although most existing results rely heavily on comparison-based techniques.

Despite the theoretical importance of h -stability, its application to linear continuous-time systems via Lyapunov-Krasovskii functionals (LKFs) remains relatively limited in the existing literature. In particular, most available results are confined to standard state-space systems [7], while only a few attempts have addressed more general system structures. For the class of singular systems, which inherently involve algebraic constraints and may exhibit impulsive behaviors, the investigation of h -stability is even more scarce. To the best of our knowledge, a comprehensive framework based on LKFs for analyzing h -stability in linear continuous-time singular systems remains largely underexplored. This reveals a significant research gap, especially considering that singular systems arise naturally in many practical applications such as electrical networks, constrained mechanical systems, and economic models. Therefore, extending the h -stability framework to singular systems using appropriate Lyapunov-Krasovskii functionals is both

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theoretically meaningful and practically relevant. Motivated by this gap, the present study aims to develop a new methodology to address both the analysis and syn-thesis aspects of such systems. Specifically, we employ the Lyapunov-Krasovskii functional framework to derive sufficient conditions-expressed in terms of linear matrix inequalities (LMIs)-that not only guarantee h -stability but also facilitate controller design for stabilizing the system. This dual focus contributes a novel perspective to the study of h -stability in the context of continuous-time dynamics.

2. Results

We examine linear continuous-time control systems governed by the following dynamic equation

$$E\dot{x}(t) = Ax(t) + Bu(t), x(t_0) = x_0 \quad \forall t \geq t_0 \geq 0 \quad (1)$$

where, $x(t) \in \mathbb{R}^n$ denotes the state vector, while $u(t) \in \mathbb{R}^m$ represents the control input. The matrices $E, A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{n \times m}$ are known constant matrices with compatible dimensions. It is assumed that the matrix E is singular, satisfying $\text{rank}(E) = r < n$.

The objective of this study is to construct a state-feedback control (S-FDC) law such that the resulting closed-loop system (CLD-LS) associated with (1) achieves h -stability. To this end, we consider a controller of the form

$$u(t) = Kx(t), \quad (2)$$

where K is the feedback gain matrix to be designed.

Substituting (2) into (1) leads to the following CLD-LS:

$$\begin{cases} E\dot{x}(t) = (A+BK)x(t), \\ x(t_0) = x_0 \end{cases} \quad (3)$$

Let $x(t) = x(t, t_0, x_0)$ denote the unique solution of the closed-loop system (3) corresponding to the initial condition $x_0 \in \mathbb{R}^n$ at time $t = t_0$.

Definition 2.1 ([8,9]). The pair matrix (E, A) is said to be regular if $\det(sE - A)$ is not identically zero. The pair matrix (E, A) is said to be impulse-free if $\deg(\det(sE - A)) = \text{rank}(E)$.

Definition 2.2 ([8,9]). The system (1) with $u(t) = 0$ is said to be regular and impulse-free, if the pair matrix (E, A) is regular and impulse-free.

The following are definitions of various well-established stability concepts, as presented in [6].

Definition 2.3 ([8,9]). The CLD-LS (3) is said to be uniformly stable if for each $\varepsilon > 0$ there is $\delta = \delta(\varepsilon) > 0$ such that

$$\|x(t_0)\| < \delta \Rightarrow \|x(t)\| < \varepsilon, \quad \forall t \geq t_0 \geq 0 \quad (4)$$

Definition 2.4 ([10]). The CLD-LS (3) is said to be exponentially stable, if there exist positive constant numbers N and α such that for each $x_0 \in \mathbb{R}^n$ and for any $t_0 \in \mathbb{R}^+$.

$$\|x(t)\| < N\|x_0\|e^{-\alpha(t-t_0)}, \forall t \geq t_0 \geq 0 \quad (5)$$

Definition 2.5 ([4,7]). The CLD-LS (3) is called globally h -stable if there exist $N \geq 1$ and a non-increasing differentiable function $h: t_0, +\infty) \rightarrow t_0, +\infty)$ such that for each $x_0 \in \mathbb{R}^n$, we have

$$\|x(t)\| < N\|x_0\|h(t)h(t_0)^{-1}, \forall t \geq t_0 \geq 0, \quad (6)$$

where

$$h(t)^{-1} = \frac{1}{h(t)}.$$

Theorem 2.6. For given scalars $\mu > 0$, if there is a non-increasing differentiable function $h: t_0, +\infty) \rightarrow (0, +\infty)$ subject $-h^{-1}(t)\dot{h}(t) \leq \mu$, a symmetric positive definite matrix S and any matrix Z such that the following matrix inequalities holds

$$SE = E^T S \geq 0, \quad (7)$$

$$2\mu SE + A^T S + SA + ZL_E^T A + A^T L_E Z^T < 0, \quad (8)$$

Where, L_E is a constant matrix satisfying $E^T L_E = 0$ and $\text{rank}(L_E) = n - r$. Then, the system (1) with $u(t) = 0$ is regular, impulse-free and globally h -stable.

Proof. Since $\text{rank}(E) = r < n$, these exist non-singular matrices M and N such that

$$\tilde{E} := MEN = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}, \tilde{A} := MAN = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}. \quad (9)$$

Let

$$\Omega = N^T S M^{-1} = \begin{bmatrix} \Omega_{11} & \Omega_{12} \\ \Omega_{21} & \Omega_{22} \end{bmatrix}, \tilde{Z} = N^T Z = \begin{bmatrix} Z_{11} \\ Z_{21} \end{bmatrix}. \quad (10)$$

From (7), we have

$$N^T S M^{-1} M E N = N^T E^T M^T M^{-T} S N \geq 0. \quad (11)$$

Combining (9) - (11) we obtain

$$\begin{bmatrix} \Omega_{11} & \Omega_{12} \\ \Omega_{21} & \Omega_{22} \end{bmatrix} \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \Omega_{11} & \Omega_{12} \\ \Omega_{21} & \Omega_{22} \end{bmatrix} \geq 0 \quad (12)$$

Combining (12), we obtain $\Omega_{11} = \Omega_{11}^T \geq 0$ and $\Omega_{21} = 0$. Hence S is a symmetric positive definite matrix, thus $\Omega_{11} = \Omega_{11}^T > 0$.

From $\text{rank}(L_E) = n - r$ and $E^T L_E = 0$, it follows that

$$N^T E^T L_E = N^T E^T M^T M^{-T} L_E = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} M^{-T} L_E = 0. \quad (13)$$

Thus, we can parameterize as $L_E = M^T \begin{bmatrix} 0 \\ \Phi \end{bmatrix}$. By pre- and post-multiplying both sides of (8) with N^T and its transpose, we obtain

$$2\mu N^T S E N + N^T A^T S N + N^T S A N + N^T Z \begin{bmatrix} 0 \\ \Phi \end{bmatrix}^T M A N + N^T A^T M^T \begin{bmatrix} 0 \\ \Phi \end{bmatrix} Z^T N < 0, \quad (14)$$

That is

$$2\mu \Omega \tilde{E} + \tilde{A}^T \Omega + \Omega^T \tilde{A} + \begin{bmatrix} Z_{11} \\ Z_{21} \end{bmatrix} \begin{bmatrix} 0 \\ \Phi \end{bmatrix}^T \tilde{A} + \tilde{A}^T \begin{bmatrix} 0 \\ \Phi \end{bmatrix} \begin{bmatrix} Z_{11} \\ Z_{21} \end{bmatrix}^T < 0. \quad (15)$$

From (15), we obtain

$$(\Omega_{22} + Z_{21} \Phi^T) A_{22} + A_{22}^T (\Omega_{22}^T + Z_{21} \Phi^T)^T < 0. \quad (16)$$

which implies that A_{22} is nonsingular. Therefore, the pair (E, A) is regular and impulse-free (see [8, 9]). Then, system (1) in the absence of the control input, i.e., when $u(t) = 0$ is regular and impulse-free.

A Lyapunov–Krasovskii functional is introduced in the following form

$$V(t) = h^{-2}(t) h^2(t_0) x^T(t) S E x(t). \quad (17)$$

First, the derivative of $V(t)$ with respect to time, evaluated along the trajectories of system (1), can be expressed as follows:

$$\begin{aligned}\dot{V}(t) &= -2h^{-1}(t)\dot{h}(t)\left(h^{-2}(t)h^2(t_0)x^\top(t)SEx(t)\right) \\ &\quad + 2h^{-2}(t)h^2(t_0)x^\top(t)SE\dot{x}(t) \\ &\leq 2\mu\left(h^{-2}(t)h^2(t_0)x^\top(t)SEx(t)\right) \\ &\quad + 2h^{-2}(t)h^2(t_0)x^\top(t)SAx(t) \\ &\leq h^{-2}(t)h^2(t_0)(2\mu x^\top(t)SEx(t) + 2x^\top(t)SAx(t)) \\ &\leq h^{-2}(t)h^2(t_0)x^\top(t)(2\mu SE + SA + A^\top S)x(t).\end{aligned}\quad (18)$$

Moreover, since $E^\top L = 0$, from first equation in (1), we have

$$2h^{-2}(t)h^2(t_0)x^\top(t)ZL_E^\top E\dot{x}(t) = 2h^{-2}(t)h^2(t_0)x^\top(t)ZL_E^\top Ax(t) = 0. \quad (19)$$

Combining (18) and (19), we get

$$\dot{V}(t) \leq h^{-2}(t)h^2(t_0)x^\top(t)(2\mu SE + A^\top S + SA + ZL_E^\top A + A^\top L_E Z^\top)x(t). \quad (20)$$

From (8) and (20), it follows that $\dot{V}(t) \leq 0$. Hence, we conclude that

$$V(t) \leq V(t_0), \forall t \geq t_0 \geq 0. \quad (21)$$

It follows that

$$h^{-2}(t)h^2(t_0)x^\top(t)SEx(t) \leq x^\top(t_0)SEx(t_0) \leq \lambda_{\max}(SE) \|x_0\|^2. \quad (22)$$

Moreover,

$$V(t) = h^{-2}(t)h^2(t_0)x^\top(t)SEx(t) \geq \lambda_{\min}(N^\top \Omega M^{-1})_{r \times r} h^{-2}(t)h^2(t_0) \|x(t)\|^2. \quad (23)$$

From (17) – (23), we have

$$\lambda_{\min}(N^\top \Omega M^{-1})_{r \times r} h^{-2}(t)h^2(t_0) \|x(t)\|^2 \leq \lambda_{\max}(SE) \|x_0\|^2. \quad (24)$$

Therefore,

$$\|x(t)\| \leq \sqrt{\frac{\lambda_{\max}(SE)}{\lambda_{\min}(N^\top \Omega M^{-1})_{r \times r}}} \|x_0\| h(t)(h(t_0))^{-1}. \quad (25)$$

which shows that system (1) is h -stable. The proof is completed.

Theorem 2.7. For given scalars $\mu > 0$, if there is a non-increasing differentiable function $h: t_0, +\infty) \rightarrow (0, +\infty)$ subject to $-h^{-1}(t)\dot{h}(t) \leq \mu$, a symmetric positive definite matrix Π and any matrix Ξ such that the following matrix inequalities holds

$$E\Pi = \Pi E^\top \geq 0, \quad (26)$$

$$2\mu E\Pi + \Pi A^\top + A\Pi + B\Xi + \Xi^\top B^\top + L_E^\top A\Pi + \Pi A^\top L_E + L_E^\top B\Xi + \Xi^\top B^\top L_E < 0. \quad (27)$$

Then, the closed-loop system(3) is regular, impulse-free and globally h -stable. The controller gain is obtained as $K = \Xi\Pi^{-1}$, where L_E is a constant matrix satisfying $E^\top L_E = 0$ and $\text{rank} L_E = n - r$.

Proof. According to Theorem 2.6, replace A with $A + BK$ and let $Z = S$. By pre- and post-multiplying both sides of (7)-(8) with S^{-1} and its transpose, respectively, we obtain

$$ES^{-1} = S^{-1}E^\top, \quad (28)$$

$$\begin{aligned}2\mu ES^{-1} + S^{-1}(A + BK)^\top + (A + BK)S^{-1} + L_E^\top(A + BK)S^{-1} \\ + S^{-1}(A + BK)^\top L_E < 0\end{aligned}\quad (29)$$

We define the following matrix transformations

$$\Pi = S^{-1}, \Xi = KS^{-1}. \quad (30)$$

Then, we get the LMIs (26) and (27). The proof is completed.

3. Numerical examples

Example 3.1. In this example, system (3) is examined under the specification of the following matrices:

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, A = \begin{bmatrix} -0.25 & 0.05 & -0.02 \\ 0.05 & 0.16 & 0.13 \\ -0.05 & 0.15 & -0.5 \end{bmatrix}, B = \begin{bmatrix} -0.05 \\ -0.02 \\ 0.01 \end{bmatrix}$$

And a time-varying function $h(t) = e^{-0.5t}$. It can be readily verified that

$$-h^{-1}(t)\dot{h}(t) = 0.5 \leq \mu = 0.5,$$

Satisfying the condition in Theorem 2.7.

By employing the LMI Toolbox available in MATLAB, a feasible matrix Π is obtained through numerical computation.

$$\Pi = \begin{bmatrix} 6.8887 & 1.1641 & 0 \\ 1.1641 & 0.2251 & 0 \\ 0 & 0 & 1.6772 \end{bmatrix},$$

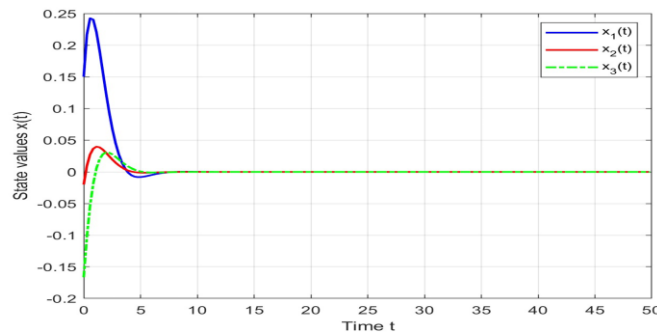


Figure 1. The state trajectories $x_1(t)$, $x_2(t)$, and $x_3(t)$ of the CLD-LS (3)

And $E = [87.3751 \quad 33.5316 \quad 1.3441]$,

which yields the state-feedback controller gain for system (3) as:

$$K = [-98.8408 \quad 659.9652 \quad 0.8014]$$

Assuming the initial conditions $x_1(0) = 0.15, x_2(0) = -0.02, x_3(0) = -0.17$, we estimate the norm of the state trajectory as:

$$\|x(t)\| \leq \sqrt{\frac{\lambda_{\max}(E\Pi)}{\lambda_{\min}(N^T\Pi^{-1}M^{-1})}} \|x_0\| e^{-\mu t} = 4.4257e^{-0.5t}, \forall t \geq 0.$$

Therefore, by Theorem 2.7, the resulting closed-loop system (3) is globally h -stable. This is illustrated in Fig. 1. Additionally, Fig. 2 highlights the instability of the open-loop system, reinforcing the effectiveness of the proposed control design.

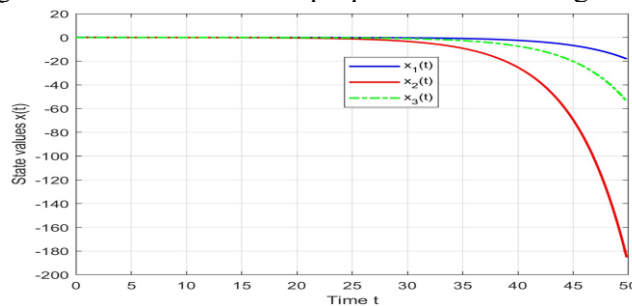


Figure 2. The state trajectories $x_1(t)$, $x_2(t)$, and $x_3(t)$ of the open-loop system

Example 3.2. In this example, we consider system (1) with $u(t) = 0$, where the system matrices are given by

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, A = \begin{bmatrix} -0.25 & 0.16 & -0.02 \\ 0.06 & -0.25 & -0.18 \\ -0.05 & 0.15 & -0.65 \end{bmatrix}.$$

The time-varying delay function is selected as $h(t) = e^{-0.15t}$. It is straightforward to verify that

$$-h^{-1}(t)\dot{h}(t) = 0.15 \leq \mu = 0.15,$$

Which satisfies the requirement stated in Theorem 2.6.

By utilizing the LMI Toolbox in MATLAB, a feasible solution is numerically obtained, yielding the matrices

$$S = \begin{bmatrix} 0.9692 & -0.0372 & 0 \\ -0.0372 & 1.3944 & 0 \\ 0 & 0 & 1.0101 \end{bmatrix},$$

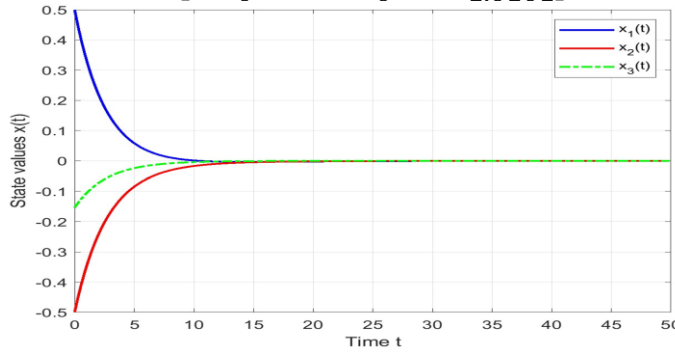


Figure 3. State trajectories $x_1(t)$, $x_2(t)$, and $x_3(t)$ of system (1) with $u(t) = 0$.

And

$$Z = \begin{bmatrix} 7.7661 & -5.2589 & -2.5785 \\ -217.5616 & 130.5497 & 86.6180 \\ -1.1960 & 0.4707 & 0.4707 \end{bmatrix}.$$

With the initial conditions $x_1(0) = 0.5$, $x_2(0) = -0.5$, and $x_3(0) = -0.15$, the state norm can be bounded as

$$\|x(t)\| \leq \sqrt{\frac{\lambda_{\max}(SE)}{\lambda_{\min}(N^T S M^{-1})}} \|x_0\| e^{-\mu t} = 1.0417 e^{-0.15t}, \forall t \geq 0.$$

Consequently, in light of Theorem 2.6, it is rigorously established that system (1) with $u(t) = 0$ is globally h -stable, and this theoretical result is further validated by the numerical simulations, as illustrated in Fig. 3.

4. Conclusion

In this study, we adopt the Lyapunov-Krasovskii functional method-an approach that generalizes exponential stability-to establish novel sufficient conditions ensuring the h -stability of closed-loop linear continuous-time singular systems. To highlight the practical relevance of the proposed criteria, several illustrative examples based on numerical simulations are included.

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