

ON THE GEVREY REGULARITY OF SOLUTIONS TO THE THREE DIMENSIONAL GENERALIZED NAVIER-STOKES EQUATIONS WITH FRACTIONAL VOIGT REGULARIZATION

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Abstract: *In this paper, we study the Gevrey regularity of solutions to the three-dimensional generalized Navier-Stokes equations with fractional Voigt regularization. Our result improves and extends previous ones.*

Keywords: *Fractional Voigt-regularization, generalized Navier-Stokes equations, fractional Laplacian, Sobolev regularity, Gevrey regularity.*

1. Introduction

In this paper, we consider the following 3D generalized Navier-Stokes equations with fractional Voigt-regularization (fGNSV) on the torus $\mathbb{T} = [-\pi, \pi]^3$ with periodic boundary conditions:

$$\begin{cases} (I + (-\alpha^2 \Delta)^r) \partial_t u + \nu (-\Delta)^\beta u + (u \cdot \nabla) u + \nabla p = f, \\ \nabla \cdot u = 0, \end{cases} \quad (1.1)$$

where $u(x, t) = (u_1(x, t), u_2(x, t), u_3(x, t))$ represents the fluid velocity vector field, $p = p(x, t)$ is the scalar pressure, $\nu > 0$ is the constant kinematic viscosity, $f(x, t)$ is the external body force, and $(-\Delta)^\beta$ is the β -fractional Laplacian. The system (1.1) was recently studied in for global well-posedness and Sobolev regularity.

In the case $\beta = 1$, (1.1) becomes the fractional Navier-Stokes-Voigt equations (fNSV):

$$\begin{cases} (I + (-\alpha^2 \Delta)^r) \partial_t u - \nu \Delta u + (u \cdot \nabla) u + \nabla p = f, \\ \nabla \cdot u = 0. \end{cases} \quad (1.2)$$

The system (1.2) was recently studied in [3] for global well-posedness and related issues. Attractors and their dimensions of (1.2) are studied in [5].

In the case $r = \beta = 1$, (1.1) becomes the Navier-Stokes-Voigt equations (NSV):

$$\begin{cases} (I - \alpha^2 \Delta) \partial_t u - \nu \Delta u + (u \cdot \nabla) u + \nabla p = f, \\ \nabla \cdot u = 0. \end{cases} \quad (1.3)$$

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The system (1.3) was first introduced by Oskolkov in [10] as a model of motion of certain linear viscoelastic fluids. The system (1.3) was also proposed by Cao, Lunasin, and Titi in [4] as a regularization, for a small value of α , of the 3D Navier-Stokes equations for the sake of direct numerical simulations. For the global well-posedness, regularity, asymptotic analysis, and applications of (1.2) and (1.3), we refer readers to [1, 2, 3, 4, 7, 8, 10, 11] and references therein for more details.

Motivated by [1, 7, 8], the main purpose of this paper is to investigate the Gevrey regularity of solutions to (1.1). The main result of the paper is the content of Theorem 3.1. The paper is organized as follows. In Section 2, we recall the preliminaries. In Section 3, we study the Gevrey regularity. In this paper, we denote by $A \lesssim B$ an estimate of the form $A \leq CB$, where C is a non-dimensional positive constant that may change from line to line.

2. Preliminaries

Since we consider the torus where the spatial variable $x \in \mathbb{T} = [-\pi, \pi]^3$ and periodic boundary conditions are assumed, we can restrict ourselves to dealing with initial data and f with vanishing spatial averages; then the solutions will also enjoy the same property. Therefore, our paper can be used as preliminary material as outlined below:

i) The velocity vectors $u : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ which are divergence-free, periodic and have zero spatial averages can be presented as follows:

$$u := \sum_{k \in J} u_k \phi_k \text{ with } u_k \in \mathbb{C}^3, u_k^* = u_{-k}, u_k \cdot k = 0 \quad \forall k \in J,$$

where $\phi_k = e^{ik \cdot x}$, $J = \mathbb{Z}^3 \setminus \{0\}$.

ii) Let $\mathcal{F}$ be the space of formal Fourier series

$$\{u := \sum_{k \in J} u_k \phi_k, u_k \in \mathbb{C}^3, \phi_k = e^{ik \cdot x}\},$$

and let $\mathcal{V}$ be the space of divergence-free trigonometric polynomials consisting of all $u \in \mathcal{F}$ such that $k \cdot u_k = 0$ for all $k \in J$ and $u_k = 0$ for all but finitely many values of $k \in J$.

iii) We define the following spaces:

$$H^s := \{u \in \mathcal{F} : u = \sum_{k \in J} u_k \phi_k, u_k \in \mathbb{C}^3, u_k^* = u_{-k}, \phi_k = e^{ik \cdot x} \text{ and } \|u\|_{H^s}^2 := \sum_{k \in J} |u_k|^2 |k|^{2s} < \infty\},$$

and

$$V^s := \{u \in \mathcal{F} : u = \sum_{k \in J} u_k \phi_k, u_k \in \mathbb{C}^3, u_k^* = u_{-k}, u_k \cdot k = 0, \phi_k = e^{ik \cdot x} \text{ and } \|u\|_{V^s}^2 := \sum_{k \in J} |u_k|^2 |k|^{2s} < \infty\},$$

for each $s \in \mathbb{R}$. These spaces are also Hilbert spaces with scalar product

$$\langle u, v \rangle_{V^s} = \sum_{k \in J} u_k \cdot v_{-k} |k|^{2s}.$$

We see that V^s is the closure of $\mathcal{V}$ in H^s with respect to the $\|\cdot\|_{H^s}$ norm. For simplicity, we use the notation $\langle \cdot, \cdot \rangle$ to denote the scalar product in V^0 and also the dual pairing of $V^s - V^{-s}$ by $\langle u, v \rangle := \sum_{k \in J} u_k \cdot v_{-k}$.

iv) Let $s \in \mathbb{R}$ and $\varepsilon > 0$. We have the following compact embedding $V^{s+\varepsilon} \subset\subset V^s$. Moreover, if $s_1 \leq s_2$ and $u \in V^{s_2}$, then $\|u\|_{V^{s_1}} \lesssim \|u\|_{V^{s_2}}$. If $s = \gamma s_1 + (1-\gamma)s_2$, $0 \leq \gamma \leq 1$, then $\|u\|_{V^s} \lesssim \|u\|_{V^{s_1}}^\gamma \|u\|_{V^{s_2}}^{1-\gamma}$.

v) We have continuous embeddings between V^s and $L^p(\mathbb{T})$ as follows:

If $0 \leq s < \frac{3}{2}$ and $\frac{1}{p} \geq \frac{1}{2} - \frac{s}{3}$, then $\|u\|_{L^p(\mathbb{T})} \lesssim \|u\|_{V^s}$, for all $u \in V^s$.

If $s = \frac{3}{2}$, then $\|u\|_{L^p(\mathbb{T})} \lesssim \|u\|_{V^s}$ for any finite p and all $u \in V^s$.

If $s > \frac{3}{2}$, then $\|u\|_{L^\infty(\mathbb{T})} \lesssim \|u\|_{V^s}$, for all $u \in V^s$.

vi) The linear operator $\Lambda = (-\Delta)^{\frac{1}{2}}$ is defined as follows:

$$\Lambda u = \sum_{k \in J} |k| u_k \phi_k \text{ with } u = \sum_{k \in J} u_k \phi_k, \phi_k = e^{ik \cdot x}.$$

We define the linear operator Λ^s by $\Lambda^s u = \sum_{k \in J} |k|^s u_k \phi_k$. Thus, $(-\Delta)^s = \Lambda^{2s}$. Since Λ^s preserves the divergence-free condition $k \cdot u_k = 0$, we can infer that Λ^s maps V^α onto $V^{\alpha-s}$. In particular, Λ^s maps V^s onto V^0 for all $s > 0$, so $D(\Lambda^s) = V^s$. It follows from the construction of Λ^s that $\|u\|_{V^s} = \|\Lambda^s u\|_{V^0}$.

Denote by P_σ the Leray-Helmholtz projection. It is the orthogonal projection from $L^2(\mathbb{T})$ onto V^0 and $P_\sigma \Lambda^s = \Lambda^s P_\sigma$.

vii) For $\sigma > 0$, $s > 0$ and the functions u obeying $\sum_{k \in J} e^{2\sigma|k|^s} |u_k|^2 < \infty$, we define $e^{\sigma \Lambda^s} u = \sum_{k \in J} e^{\sigma|k|^s} u_k \phi_k$.

We define the Gevrey class of order $s > 0$ for a given $\tau \geq 0$ as the domain of a particular class of operators parameterized by $\sigma > 0$:

$$\mathcal{D}(\Lambda^\tau e^{\sigma \Lambda^s}) := \{u \in V^\tau : \|e^{\sigma \Lambda^s} u\|_{V^\tau} < +\infty\}.$$

If $s = 1$, then $\mathcal{D}(\Lambda^\tau e^{\sigma \Lambda})$ is the space of real analytic functions (see, e.g., [6, 9]).

Now, we apply the projection operator P_σ to eliminate the pressure. Due to the periodic setting, the weak formulation (1.1) is equivalent to the following abstract formulation:

$$(I + \alpha^{2r} \Lambda^{2r}) \partial_t u + \nu \Lambda^{2\beta} u + B(u, u) = P_\sigma f. \quad (2.1)$$

where $B(u, v) := P_\sigma \{(u \cdot \nabla) v\}$.

Our system should be supplemented with L^2 initial data $u(\tau, x) = u_\tau(x)$. Let us start with the definitions of solutions for (2.1).

Definition 2.1. (Weak solution) Let ν , α , β , and r be positive real parameters. Given $f \in L^2_{loc}(\mathbb{R}; V^{-\beta})$, $u_\tau \in V^r$ and a fixed $T > \tau$. A weak solution to (2.1) on the interval $[\tau, T]$ is a function $u(t, x)$ such that

$$u \in C([\tau, T]; V^0) \cap L^\infty(\tau, T; V^r) \cap L^2(\tau, T; V^\beta), \partial_t u \in L^2(\tau, T; V^{-\eta}),$$

where $\eta = \max\{\beta, r\}$ and, given any $t > \tau$ and $\gamma = \max\{\frac{5}{2} - \eta, \eta\}$, it satisfies $u(\tau) = u_\tau$ and

$$\begin{aligned} \langle u(t), v \rangle + \alpha^{2r} \langle \Lambda^r u(t), \Lambda^r v \rangle + \nu \int_\tau^t \langle \Lambda^\beta u(s), \Lambda^\beta v \rangle ds - \int_\tau^t \langle B(u(s), v), u(s) \rangle ds \\ = \langle u_\tau, v \rangle + \int_\tau^t \langle f(s), v \rangle ds, \end{aligned} \quad (2.2)$$

for a.e. $t \in [\tau, T]$, $v \in V^\gamma$.

Definition 2.2. (Regular solution) Let ν , α , β , δ , and r be positive real parameters. Given $f \in L^2_{loc}(\mathbb{R}; V^{-\beta+\delta})$, $u_\tau \in V^{r+\delta}$ and a fixed $T > \tau$. We say that a function u is a regular solution to (2.1) if u is a weak solution to (2.1) and, in addition,

$$u \in C([\tau, T]; V^\delta) \cap L^\infty(\tau, T; V^{r+\delta}) \cap L^2(\tau, T; V^{\beta+\delta}).$$

We denote $\mathfrak{M} = \mathfrak{M}_1 \cup \mathfrak{M}_2$, and $\mathfrak{J} = \mathfrak{J}_1 \cup \mathfrak{J}_2 \cup \mathfrak{J}_3 \cup \mathfrak{J}_4$, where

$$\mathfrak{M}_1 := \{(r, \beta) \mid r \geq \beta > 0, r \geq \frac{5}{8}\},$$

$$\mathfrak{M}_2 := \{(r, \beta) \mid \beta > r > 0, \beta + r \geq \frac{5}{4}\},$$

$$\mathfrak{J}_1 := \{(r, \beta) \mid r \geq \beta > 0, r \geq \frac{5}{4}\},$$

$$\mathfrak{J}_2 := \{(r, \beta) \mid r \geq \beta > 0, \frac{5}{4} > r \geq 1, 2r + \beta \geq \frac{5}{2}\},$$

$$\mathfrak{J}_3 := \{(r, \beta) \mid \beta > r > 0, \beta \geq \frac{5}{4}\},$$

$$\mathfrak{J}_4 := \{(r, \beta) \mid \beta > r > 0, \frac{5}{4} > \beta \geq 1, 2\beta + r \geq \frac{5}{2}\},$$

Let us recall the existence, uniqueness, and Sobolev regularity of weak solutions to (2.1).

Theorem 2.1. Let ν , α , β , δ , and r be real positive parameters. Assume that $u_\tau \in V^r$ and $f \in L^2_{loc}(\mathbb{R}; V^{-\beta})$.

(i) If $(r, \beta) \in \mathfrak{M}$, then the system (2.1) possesses a global weak solution obeying Definition 2.1 with the initial condition u_τ . Furthermore, if $(r, \beta) \in \mathfrak{J}$, then the global weak solution is unique and depends continuously on the initial data.

(ii) If $(r, \beta) \in \mathfrak{J}$, $u_\tau \in V^{r+\delta}$ and $f \in L^2_{loc}(\mathbb{R}; V^{-\beta+\delta})$, then the system (2.1) possesses a unique global regular solution obeying Definition 2.2 with the initial condition u_τ .

We also recall the following important lemma (see, e.g., [1, Lemma 4.1]).

Lemma 2.1. Let $u \in \mathcal{D}(\Lambda^{\delta+\frac{1}{2s}} e^{\sigma\Lambda^{\frac{1}{s}}})$ and $v \in \mathcal{D}(\Lambda^{\delta+\frac{1}{2s}+\frac{1}{2}} e^{\sigma\Lambda^{\frac{1}{s}}})$, where $s \geq 1$, and $\delta > \frac{5}{2} + \frac{1}{2s}$. We have

$$\begin{aligned} |\langle \Lambda^\delta e^{\sigma\Lambda^{\frac{1}{s}}} B(u, v), \Lambda^\delta e^{\sigma\Lambda^{\frac{1}{s}}} v \rangle| &\lesssim \| \Lambda^\delta u \|_{L^2(\mathbb{T})} \| \Lambda^\delta v \|_{L^2(\mathbb{T})}^2 \\ &\quad + \sigma \left\{ \| \Lambda^\delta e^{\sigma\Lambda^{\frac{1}{s}}} u \|_{L^2(\mathbb{T})} \| \Lambda^{\delta+\frac{1}{2s}} e^{\sigma\Lambda^{\frac{1}{s}}} v \|_{L^2(\mathbb{T})}^2 \right. \\ &\quad \left. + \| \Lambda^\delta e^{\sigma\Lambda^{\frac{1}{s}}} v \|_{L^2(\mathbb{T})} \| \Lambda^{\delta+\frac{1}{2s}} e^{\sigma\Lambda^{\frac{1}{s}}} u \|_{L^2(\mathbb{T})} \| \Lambda^{\delta+\frac{1}{2s}} e^{\sigma\Lambda^{\frac{1}{s}}} v \|_{L^2(\mathbb{T})} \right\}. \end{aligned}$$

3. Gevrey Regularity

In this section, we will study the Gevrey regularity of solutions to (2.1). The main result is the content of the following theorem.

Theorem 3.1. Let $\nu, \alpha, \beta, \delta, r, s$ be real positive parameters such that $s \geq 1$, $\delta > \frac{5}{2} + \frac{1}{2s}$, and let $\sigma_0 \geq 0$. Assume that $(r, \beta) \in \mathfrak{J}$, $u_0 \in \mathcal{D}(\Lambda^{r+\delta} e^{\sigma_0\Lambda^{\frac{1}{s}}})$ and

$f \in L^2_{loc}(\mathbb{R}; \mathcal{D}(\Lambda^\delta e^{\sigma_0\Lambda^{\frac{1}{s}}}))$, then there exists a unique solution u of (2.1) on $[0, T]$ satisfying

$$\begin{aligned} &\| \Lambda^\delta e^{\sigma(t)\Lambda^{\frac{1}{s}}} u(t) \|_{L^2(\mathbb{T})}^2 + \alpha^{2r} \| \Lambda^\delta e^{\sigma(t)\Lambda^{\frac{1}{s}}} \Lambda^r u(t) \|_{L^2(\mathbb{T})}^2 + \nu \int_0^t \| \Lambda^\delta e^{\sigma(\eta)\Lambda^{\frac{1}{s}}} \Lambda^\beta u(\eta) \|_{L^2(\mathbb{T})}^2 d\eta \\ &\leq \| \Lambda^\delta e^{\sigma_0\Lambda^{\frac{1}{s}}} u_0 \|_{L^2(\mathbb{T})}^2 + \alpha^{2r} \| \Lambda^\delta e^{\sigma_0\Lambda^{\frac{1}{s}}} \Lambda^r u_0 \|_{L^2(\mathbb{T})}^2 \\ &\quad + 2c_1 \int_0^t \kappa^3(\eta) d\eta + 2c_3 \int_0^t \| \Lambda^\delta e^{\sigma_0\Lambda^{\frac{1}{s}}} f(\eta) \|_{L^2(\mathbb{T})}^2 d\eta, \end{aligned}$$

for all $t \in [0, T]$ and some positive constants c_1, c_3 , where $\kappa(t)$ and $\sigma(t)$ are determined by (3.4) and (3.6), respectively.

Proof. Obviously, we have $\mathcal{D}(\Lambda^{r+\delta} e^{\sigma_0\Lambda^{\frac{1}{s}}}) \subset V^{r+\delta}$. Therefore, the conclusions of Theorem 2.1 hold on $[0, T]$. This implies that the case $\sigma_0 = 0$ is already covered by Theorem 2.1, so we can assume that $\sigma_0 > 0$. Applying $\Lambda^\delta e^{\sigma\Lambda^{\frac{1}{s}}}$ to both sides of (2.1) and using the following identity

$$\Lambda^\delta e^{\sigma\Lambda^{\frac{1}{s}}} \partial_t u = \partial_t (\Lambda^\delta e^{\sigma\Lambda^{\frac{1}{s}}} u) - \dot{\sigma} \Lambda^{\delta+\frac{1}{s}} e^{\sigma\Lambda^{\frac{1}{s}}} u,$$

we get

$$\begin{aligned} & \partial_t [\Lambda^\delta e^{\sigma\Lambda^{\frac{1}{s}}} (u + \alpha^{2r} \Lambda^{2r} u)] + \nu \Lambda^\delta e^{\sigma\Lambda^{\frac{1}{s}}} \Lambda^{2\beta} u + \Lambda^\delta e^{\sigma\Lambda^{\frac{1}{s}}} B(u, u) \\ &= \dot{\sigma} \Lambda^{\delta+\frac{1}{s}} e^{\sigma\Lambda^{\frac{1}{s}}} (u + \alpha^{2r} \Lambda^{2r} u) + \Lambda^\delta e^{\sigma\Lambda^{\frac{1}{s}}} P_\sigma f. \end{aligned} \quad (3.1)$$

We take the L^2 -scalar product of (3.1) with itself $\Lambda^\delta e^{\sigma\Lambda^{\frac{1}{s}}} u$, we get

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|\Lambda^\delta e^{\sigma\Lambda^{\frac{1}{s}}} u\|_{L^2(\mathbb{T})}^2 + \frac{\alpha^{2r}}{2} \frac{d}{dt} \|\Lambda^\delta e^{\sigma\Lambda^{\frac{1}{s}}} \Lambda^r u\|_{L^2(\mathbb{T})}^2 + \nu \|\Lambda^\delta e^{\sigma\Lambda^{\frac{1}{s}}} \Lambda^\beta u\|_{L^2(\mathbb{T})}^2 \\ &= \dot{\sigma} \left\{ \|\Lambda^{\delta+\frac{1}{2s}} e^{\sigma\Lambda^{\frac{1}{s}}} u\|_{L^2(\mathbb{T})}^2 + \alpha^{2r} \|\Lambda^{\delta+\frac{1}{2s}} e^{\sigma\Lambda^{\frac{1}{s}}} \Lambda^r u\|_{L^2(\mathbb{T})}^2 \right\} \\ & \quad - \langle \Lambda^\delta e^{\sigma\Lambda^{\frac{1}{s}}} B(u, u), \Lambda^\delta e^{\sigma\Lambda^{\frac{1}{s}}} u \rangle \\ & \quad + \langle \Lambda^\delta e^{\sigma\Lambda^{\frac{1}{s}}} P_\sigma f, \Lambda^\delta e^{\sigma\Lambda^{\frac{1}{s}}} u \rangle. \end{aligned} \quad (3.2)$$

Using the Cauchy-Schwarz inequality and Lemma 2.1, we deduce from (3.2) that

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|\Lambda^\delta e^{\sigma\Lambda^{\frac{1}{s}}} u\|_{L^2(\mathbb{T})}^2 + \frac{\alpha^{2r}}{2} \frac{d}{dt} \|\Lambda^\delta e^{\sigma\Lambda^{\frac{1}{s}}} \Lambda^r u\|_{L^2(\mathbb{T})}^2 + \frac{\nu}{2} \|\Lambda^\delta e^{\sigma\Lambda^{\frac{1}{s}}} \Lambda^\beta u\|_{L^2(\mathbb{T})}^2 \\ & \leq c_1 \|\Lambda^\delta u\|_{L^2(\mathbb{T})}^3 + \dot{\sigma} \alpha^{2r} \|\Lambda^{\delta+\frac{1}{2s}} e^{\sigma\Lambda^{\frac{1}{s}}} \Lambda^r u\|_{L^2(\mathbb{T})}^2 \\ & \quad + \left\{ \dot{\sigma} + c_2 \sigma \|\Lambda^\delta e^{\sigma\Lambda^{\frac{1}{s}}} u\|_{L^2(\mathbb{T})} \right\} \|\Lambda^{\delta+\frac{1}{2s}} e^{\sigma\Lambda^{\frac{1}{s}}} u\|_{L^2(\mathbb{T})}^2 + c_3 \|\Lambda^\delta e^{\sigma\Lambda^{\frac{1}{s}}} f\|_{L^2(\mathbb{T})}^2, \end{aligned} \quad (3.3)$$

for some positive constants c_1 , c_2 , and c_3 .

We infer from the Sobolev regularity that

$$\|\Lambda^\delta u(t)\|_{L^2(\mathbb{T})} \leq \kappa(t), \quad \forall t \in [0, T]. \quad (3.4)$$

We define the following functions:

$$\begin{aligned} h(t) &:= \left\{ 1 + \|\Lambda^\delta e^{\sigma_0\Lambda^{\frac{1}{s}}} u_0\|_{L^2(\mathbb{T})}^2 + \alpha^{2r} \|\Lambda^\delta e^{\sigma_0\Lambda^{\frac{1}{s}}} \Lambda^r u_0\|_{L^2(\mathbb{T})}^2 \right. \\ & \quad \left. + 2c_1 \int_0^t \kappa^3(\eta) d\eta + 2c_3 \int_0^t \|\Lambda^\delta e^{\sigma_0\Lambda^{\frac{1}{s}}} f(\eta)\|_{L^2(\mathbb{T})}^2 d\eta \right\}^{\frac{1}{2}}, \end{aligned} \quad (3.5)$$

and

$$\sigma(t) := \sigma_0 e^{-c_2 \int_0^t h(\eta) d\eta}. \quad (3.6)$$

We deduce from (3.5) and (3.6) that $h(t) > 0$, and $\sigma(t) > 0$ for all $t \in [0, T]$. Therefore, we have

$$\dot{\sigma}(t) = -c_2 h(t) \sigma(t) < 0, \quad \forall t \in [0, T]. \quad (3.7)$$

Now we will prove that

$$\| \Lambda^\delta e^{\sigma(t)\Lambda^{\frac{1}{s}}} u(t) \|_{L^2(\mathbb{T})} < h(t), \forall t \in [0, T]. \quad (3.8)$$

In the case of $t = 0$, (3.8) holds because of (3.5). Setting

$$t^* := \sup\{\theta > 0, \theta \in [0, T] \text{ such that (3.8) holds on } [0, \theta]\}.$$

Using (3.7), (3.8), and integrating (3.3) on $[0, t^*]$ implies that

$$\begin{aligned} & \| \Lambda^\delta e^{\sigma(t^*)\Lambda^{\frac{1}{s}}} u(t^*) \|_{L^2(\mathbb{T})}^2 + \alpha^{2r} \| \Lambda^\delta e^{\sigma(t^*)\Lambda^{\frac{1}{s}}} \Lambda^r u(t^*) \|_{L^2(\mathbb{T})}^2 \\ & + \nu \int_0^{t^*} \| \Lambda^\delta e^{\sigma(\eta)\Lambda^{\frac{1}{s}}} \Lambda^\beta u(\eta) \|_{L^2(\mathbb{T})}^2 d\eta \\ & \leq \| \Lambda^\delta e^{\sigma_0\Lambda^{\frac{1}{s}}} u_0 \|_{L^2(\mathbb{T})}^2 + \alpha^{2r} \| \Lambda^\delta e^{\sigma_0\Lambda^{\frac{1}{s}}} \Lambda^r u_0 \|_{L^2(\mathbb{T})}^2 \\ & + 2c_1 \int_0^{t^*} \| \Lambda^\delta u(\eta) \|_{L^2(\mathbb{T})}^3 d\eta + 2c_3 \int_0^{t^*} \| \Lambda^\delta e^{\sigma_0\Lambda^{\frac{1}{s}}} f(\eta) \|_{L^2(\mathbb{T})}^2 d\eta < h^2(t^*). \end{aligned} \quad (3.9)$$

We infer from (3.9) that (3.8) holds for $t = t^*$. If $t^* = T$, we have the result. If $t^* < T$, the strict inequality (3.8) holds for $t \in [0, t^*]$, and this can be extended in time beyond t^* , contradicting the definition of t^* . Hence, we claim that the strict inequality (3.8) holds for $t \in [0, T]$. Therefore, we have

$$\dot{\sigma} + c_2 \sigma \| \Lambda^\delta e^{\sigma\Lambda^{\frac{1}{s}}} u \|_{L^2(\mathbb{T})} < 0, \forall t \in [0, T]. \quad (3.10)$$

We deduce from (3.3), (3.7), and (3.10) that

$$\begin{aligned} & \| \Lambda^\delta e^{\sigma(t)\Lambda^{\frac{1}{s}}} u(t) \|_{L^2(\mathbb{T})}^2 + \alpha^{2r} \| \Lambda^\delta e^{\sigma(t)\Lambda^{\frac{1}{s}}} \Lambda^r u(t) \|_{L^2(\mathbb{T})}^2 \\ & + \nu \int_0^t \| \Lambda^\delta e^{\sigma(\eta)\Lambda^{\frac{1}{s}}} \Lambda^\beta u(\eta) \|_{L^2(\mathbb{T})}^2 d\eta \\ & \leq \| \Lambda^\delta e^{\sigma_0\Lambda^{\frac{1}{s}}} u_0 \|_{L^2(\mathbb{T})}^2 + \alpha^{2r} \| \Lambda^\delta e^{\sigma_0\Lambda^{\frac{1}{s}}} \Lambda^r u_0 \|_{L^2(\mathbb{T})}^2 \\ & + 2c_1 \int_0^t \kappa^3(\eta) d\eta + 2c_3 \int_0^t \| \Lambda^\delta e^{\sigma_0\Lambda^{\frac{1}{s}}} f(\eta) \|_{L^2(\mathbb{T})}^2 d\eta. \end{aligned} \quad (3.11)$$

The proof of Theorem 3.1 is completed using (3.11) and the standard Faedo-Galerkin approximation.

Remark 3.1. The result of Theorem 3.1 can be extended to any interval $[\tau, T]$. Moreover, in the case of $s = 1$, Theorem 3.1 implies that the solutions of (2.1) are analytic on $[0, T]$.

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