

ASYMPTOTIC PERIODIC SOLUTIONS TO A CLASS OF NON-DENSELY DEFINED DIFFERENTIAL EQUATIONS WITH INFINITE DELAY

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Received: 23 May 2026/ Accepted: 20 July 2026/ Published: 30 July 2026

<https://doi.org/10.70117/hdujs.E11.7.2026.1013>

Abstract: *In this paper, we prove a Massera-type theorem for the asymptotic periodicity of bounded solutions to non-densely defined differential equations with infinite delay, using the circular spectral theory of functions.*

Keywords: *Non-densely, infinite delay, asymptotic periodicity.*

1. Introduction

The study of periodicity of solutions is a fundamental problem in the qualitative theory of evolution equations. Periodic solutions naturally arise in many applied models where the external forcing or the underlying dynamics exhibit periodic behavior, such as in population dynamics, control theory, and heat conduction processes. Understanding the existence, uniqueness, and structure of such solutions plays a central role in the long-term analysis of dynamical systems. For evolution equations governed by densely defined operators, several powerful tools have been developed. Classical approaches include the fixed point method [6] techniques based on ultimate boundedness of solutions and compactness of the Poincaré map under compact embeddings [2,14], the spectral theory of functions [18,19], and ergodic methods [11]. These results provide a solid theoretical foundation and cover a wide range of important cases. However, as pointed out in [4], there are many situations in which the operator involved is not densely defined. This case is significantly more delicate, since many classical results rely heavily on density of the domain in the underlying Banach space. A simple but illustrative example is the one dimensional heat equation with Dirichlet boundary conditions on $[0, \pi]$. If we consider the Banach space $C([0, \pi], \mathbb{R})$ with the sup-norm and define

$$A = \frac{\partial^2}{\partial x^2}, \quad \text{with} \quad D(A) = \{u \in C^2([0, \pi], \mathbb{R}) : u(0) = u(\pi) = 0\}$$

then $D(A)$ is not dense in $C([0, \pi], \mathbb{R})$ since

$$\overline{D(A)} = \{u \in C([0, \pi], \mathbb{R}) : u(0) = u(\pi) = 0\} \neq C([0, \pi], \mathbb{R}).$$

Thus, standard semigroup techniques cannot be directly applied in this framework. Motivated by such difficulties, in the past two decades researchers have developed methods to study periodic solutions for non-densely defined evolution equations. Important contributions can be found in [3,5,6,7] where new techniques were introduced to overcome the lack of density and to extend the classical theory.

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Recently, L.A. Minh and N.N. Vien [17] employed the framework of the circular spectrum of functions to prove the existence of asymptotically periodic solutions for non-densely defined evolution equations. Inspired by this line of research, in the present paper we establish a Massera-type theorem for non-densely defined differential equations with infinite delay. More precisely, we show that the asymptotic periodicity of bounded solutions inherits the circular spectrum structure of the external forcing term for a class of differential equations of the abstract form

$$\frac{du(t)}{dt} = Au(t) + L(t)u_t + f(t), \quad u \in X, t \geq s; t, s \in \mathbb{R} \quad (1.1)$$

where $A: D(A) \subset X \rightarrow X$ is a non-densely defined linear operator on a Banach space X which satisfies the Hille - Yosida condition:

($\mathbf{M}_1$): there exist $M_0 \geq 1$ and $\omega_0 \in \mathbb{R}$ such that $(\omega_0, +\infty) \subset \rho(A)$ and

$$|R(\xi, A)^n| \leq \frac{M_0}{(\xi - \omega_0)^n}, \text{ for } n \in \mathbb{N} \text{ and } \xi > \omega_0,$$

where $\rho(A)$ is the resolvent set of A and $R(\xi, A) = (\xi - A)^{-1}$; $L(t): \mathcal{B} \rightarrow X$ is a bounded linear operator such that

($\mathbf{M}_2$): For each $\phi \in \mathcal{B}$, the mapping $t \mapsto L(t)$ is continuous.

To conclude this introduction, we provide a brief outline of the paper. In Section 2, we introduce the main notations and recall the definitions and fundamental properties of the circular spectrum of functions, together with the representation of mild solutions. Section 3 contains the main result of the paper, namely, a theorem on the asymptotic periodicity of bounded solutions to non-densely defined differential equations with infinite delay of the form (1.1).

2. Preliminaries

2.1. Notations

Here and subsequently, $\mathbb{N}, \mathbb{R}, \mathbb{C}$ stand for the set of natural, real, and complex numbers, respectively. Let us denote by $L(X)$ the Banach space of all continuous linear operators on a Banach space X endowed with the usual operator norm. For a general linear operator T , we denote by $\sigma(T)$ the spectrum of T and define $\rho(T) := \mathbb{C} \setminus \sigma(T)$. Moreover, we denote by $BUC(\mathbb{R}, X), BC(\mathbb{R}, X)$ the spaces of all X -valued bounded uniformly continuous functions and bounded continuous functions on $\mathbb{R}$, respectively. We define $C_0(\mathbb{R}, X) := \{g \in BUC(\mathbb{R}, X) : \lim_{t \rightarrow \infty} g(t) = 0\}$. Finally, we will use the symbol Γ to denote the unit circle in the complex plane.

2.2. Phase spaces

We now turn to the axiomatic definition of phase space which was introduced by Hale and Kato in [8] and follow the terminology used in [10].

Let $(\mathcal{B}, |\cdot|_{\mathcal{B}})$ be the Banach space of all X -valued functions defined on $(-\infty, 0]$ which satisfy the following axiom.

(A) If $x: (-\infty, a] \mapsto X$ is continuous on $[\sigma, a]$ with $x_{\sigma} \in \mathcal{B}$ for some $\sigma < a$ then

(i) x_t is in $\mathcal{B}$ for all $t \in [\sigma, a]$;

(ii) $t \mapsto x_t$ is continuous on $[\sigma, a]$;

(iii) $N|x(t)| \leq |x_t|_{\mathcal{B}} \leq K(t - \sigma) \sup_{\sigma \leq s \leq t} |x(s)| + M(t - \sigma)|x_{\sigma}|_{\mathcal{B}}, \forall t \in [\sigma, a]$,

where $N > 0$ is a constant, $K: \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is a continuous function and $M: \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is a locally bounded function.

Lemma 2.1 ([10]). Assume that the space $\mathcal{B}$ satisfies the axiom (A). Then, the following assertions hold:

- (i) For all $\phi \in \mathcal{B}$, we have that $|\phi(0)| \leq \frac{1}{N} |\phi|_{\mathcal{B}}$.
- (ii) The space of all continuous functions from $(-\infty, 0]$ into X with compact support, which is denoted by $C_{00}((-\infty, 0]; X)$, is included in $\mathcal{B}$. Moreover, for every $\phi \in C_{00}((-\infty, 0]; X)$ with the support in $[a, 0]$, we have that

$$|\phi|_{\mathcal{B}} \leq K(-a) \sup_{\theta \leq 0} |\phi(\theta)|.$$

Definition 2.2 ([20]). The phase space $\mathcal{B}$ is called a uniform fading memory space if it satisfies axiom (A) with $K(\cdot) \equiv K$ and $M(t) \rightarrow 0$ as $t \rightarrow \infty$.

2.3. Mild solutions and their representation

Definition 2.3 ([1]). Let $\phi \in \mathcal{B}$. We say that a function $u: \mathbb{R} \rightarrow X$ is a mild solution of Equation (1.1) in $\mathbb{R}$ if the following statements hold:

- (i) u is continuous on $[\sigma, +\infty)$,
- (ii) $\int_{\sigma}^t u(s) ds \in D(A)$ for $t \geq \sigma$,
- (iii) $u(t) = \phi(0) + A\left(\int_{\sigma}^t u(s) ds\right) + \int_{\sigma}^t (L(s)u_s + f(s)) ds$ for $t \geq \sigma$,
- (iv) $u_{\sigma} = \phi$.

Note that, (i) and (ii) imply that $u(t) \in X_0 = \overline{D(A)}$, for any $t \geq \sigma$. Particularly, $\phi(0) \in X_0$. Therefore, the phase space associated with Equation (1.1) is

$$\mathcal{B}_A := \{\phi \in \mathcal{B}: \phi(0) \in X_0\}.$$

According to Hille-Yosida Theorem, see [13], we conclude that, if A satisfies the condition $(\mathbf{M}_1)$, then the part $(A_0, D(A_0))$ of A in X_0 , which is defined by

$$\begin{cases} D(A_0) = \{x \in D(A): Ax \in X_0\} \\ A_0 x = Ax. \end{cases}$$

generates a strongly continuous semigroup $(T(t))_{t \geq 0}$ on X_0 . Moreover,

$$|T(t)|_{\mathcal{L}(\overline{D(A)})} \leq M_0 e^{\omega_0 t} \quad \text{for } t \geq 0.$$

Theorem 2.4 ([1]). Let conditions $(\mathbf{M}_1)$ and $(\mathbf{M}_2)$ be satisfied. Then, for each $\phi \in \mathcal{B}_A$, Equation (1.1) has a unique global mild solution u satisfying that

$$u(t) = T(t - \sigma)\phi(0) + \lim_{\lambda \rightarrow +\infty} \int_{\sigma}^t T(t - s) B_{\lambda} (L(s)u_s + f(s)) ds \quad \text{for } t \geq \sigma$$

where $B_{\lambda} = \lambda R(\lambda, A)$ for $\lambda > \omega_0$.

In the next parts of this paper, the mild solutions of Equation (1.1) are denoted by $u(\cdot, \sigma, \phi, L, f)$. For each $t \geq s$, we define the operator $U(t, s)$, for $\phi \in \mathcal{B}_A$, by

$$U(t, s)\phi = u_t(\cdot, s, \phi, L, 0).$$

Without loss of generality, we can assume that $\omega_0 > 0$. Moreover, we make the following assumption.

(M₃): The operator $L(\cdot)$ is 1-periodic.

Proposition 2.5 ([1]). *Let $(\mathbf{M}_1) - (\mathbf{M}_3)$ be satisfied. Then, $(U(t, s))_{t \geq s \geq 0}$ is a 1-periodic strongly continuous evolutionary process on $\mathcal{B}_A$ which satisfies the following properties:*

- (i) $U(t, s) \in L(X_0)$ for all $t \geq s \geq 0$;
- (ii) $U(t, t) = I$, for every $t \in \mathbb{R}$;
- (iii) $U(t, s)U(s, r) = U(t, r)$, for all $t \geq s \geq r$;
- (iv) $U(t + 1, s + 1) = U(t, s)$ for all $t \geq s \geq 0$;
- (v) *The function $(t, s, x) \mapsto U(t, s)x$ is continuous in (t, s, x) .*

Now, for $n > \omega_0$ and for $x \in X$, we define the linear operator $Y^n: X \rightarrow \mathcal{B}$ by

$$Y^n x(\theta) = \begin{cases} (n\theta + 1)B_n x & \text{for } \frac{-1}{n} \leq \theta \leq 0, \\ 0 & \text{for } \theta < \frac{-1}{n}. \end{cases}$$

According to [1], there exists $\tilde{M} > 0$ such that

$$|Y^n x|_{\mathcal{B}} \leq \tilde{M}K(-1)||x||, \quad \forall x \in X.$$

Since, for any $x \in X$, $Y^n x(0) = B_n x \in X_0$, $Y^n x \in \mathcal{B}_A$.

Proposition 2.6 ([1]). *Let assumptions $(\mathbf{M}_1)$ and $(\mathbf{M}_2)$ hold. Then, for any $s \in \mathbb{R}$ and $\phi \in \mathcal{B}_A$, the unique integral solution $u(\cdot, s, \phi, L, f)$ of Equation (1.1) satisfies the following variation of constants formula in $\mathcal{B}_A$.*

$$u_t(\cdot, s, \phi, L, f) = U(t, s)\phi + \lim_{n \rightarrow +\infty} \int_s^t U(t, \xi)Y^n f(\xi) d\xi \quad \text{for } t \geq s. \quad (2.1)$$

2.4. Circular spectra of functions on the line

We consider the translation operator S in $BC(\mathbb{R}, X_0)$ defined as

$$[Sx](\xi) := x(1 + \xi), \quad \xi \in \mathbb{R}, x \in BC(\mathbb{R}, X_0).$$

Furthermore, we also consider the quotient space

$$Y := BC(\mathbb{R}, X_0)/C_0(\mathbb{R}, X_0).$$

Then, S induces an operator in Y that will be denoted by $\bar{S}$. It is well known that $\bar{S}$ is an isometry, so $\sigma(\bar{S}) \subset \Gamma$.

For each $x \in BC(\mathbb{R}, X_0)$ let us consider the complex function $[Sx](\lambda)$ in $\lambda \in \mathbb{C} \setminus \Gamma$ defined as

$$[Sx](\lambda) := R(\lambda, \bar{S})\bar{x}, \quad \lambda \in \mathbb{C} \setminus \Gamma.$$

Definition 2.7 ([16]). The *circular spectrum* of a function $x \in BC(\mathbb{R}, X_0)$ is defined to be the set of all $\xi_0 \in \Gamma$ such that $[Sx](\lambda)$ has no analytic extension into any neighborhood of ξ_0 in the complex plane. This spectrum of x is denoted by $\sigma(x)$.

The following lemma justifies the introduction of these concepts of spectra.

Lemma 2.8 ([16]). *For any $x \in BC(\mathbb{R}, X_0)$,*

$$\sigma(Qx) \subset \sigma(x)$$

provided that Q is an operator in $BC(\mathbb{R}, X_0)$ that commutes with S and leaves $C_0(\mathbb{R}, X_0)$ invariant.

3. Main Results

3.1. Asymptotic periodic functions and their spectral characterization

Definition 3.1 ([15]). A function $f \in BC(\mathbb{R}, X)$ is said to be asymptotic 1-periodic if

$$\lim_{t \rightarrow \infty} (f(t+1) - f(t)) = 0. \quad (3.1)$$

Proposition 3.2 ([15]). The following assertions are valid:

- (i) Let $x \in BC(\mathbb{R}, X_0)$. Then, $\sigma(x) = \emptyset$ if and only if $x \in C_0(\mathbb{R}, X_0)$;
- (ii) Let $p \in \mathbb{R}$ and $x \in BC(\mathbb{R}, X_0)$. Then, $\sigma(x) \subset \{e^{ip}\}$ if and only if

$$\lim_{t \rightarrow \infty} (x(t+1) - e^{ip}x(t)) = 0.$$

Lemma 3.3 ([15]). Assume that $Q(t), t \in \mathbb{R}$ is a family of bounded linear operators in X_0 that satisfies

- (i) The function $\mathbb{R} \times X_0 \ni (t, x) \mapsto Q(t)x \in X_0$ is continuous,
- (ii) $Q(t+1) = Q(t)$ for all $t \in \mathbb{R}$,
- (iii) $\sup_{0 \leq t \leq 1} |Q(t)| < \infty$.

Then, for each $x(\cdot) \in BC(\mathbb{R}, X_0)$ we have

$$\sigma(Qx(\cdot)) \subset \sigma(x(\cdot)),$$

where Q denotes the operator in $BC(\mathbb{R}, X_0)$ defined as

$$[Qx(\cdot)](t) := Q(t)x(t), \quad t \in \mathbb{R}.$$

3.2. Asymptotic periodic solution

Definition 3.4. A function $u(\cdot) \in BC(\mathbb{R}, X_0)$ is said to be an asymptotic mild solution of Equation (1.1) with initial condition $u_s = \phi \in \mathcal{B}_A$ if there exists a function $\epsilon(\cdot) \in C_0(\mathbb{R}, X)$ such that

$$u(t) = T(t-s)\phi(0) + \lim_{\lambda \rightarrow +\infty} \int_s^t T(t, \xi) B_\lambda (L(\xi)u_\xi + f(\xi) + \epsilon(\xi)) d\xi \quad (3.2)$$

for $t \geq s$.

where $B_\lambda = \lambda R(\lambda, A)$ for $\lambda > \omega_0$.

Now, for T is an operator in a Banach space X_0 , we denote $\sigma_\Gamma(T) := \sigma(T) \cap \Gamma$. We also recall the following well known result on the spectrum of the “monodromy” operator $P(t) := U(t+1, t)$ for each $t \geq 0$. When $t = 1$ we denote $P := P(1)$. In particular, $P = U(1, 0)$ if $(U(t, s))_{t \geq s \geq 0}$ is a 1-periodic process. Let us denote by $\mathcal{P}$ the operator of multiplication $u \mapsto \mathcal{P}u$ defined as $\mathcal{P}u(t) = P(t)u(t)$.

Lemma 3.5 ([9]). Let $(U(t, s))_{t \geq s \geq 0}$ be a 1-periodic process in X_0 . Then, for each $t \geq 0$

$$\sigma(P(t)) \setminus \{0\} = \sigma(P) \setminus \{0\}.$$

We now assume that

($\mathbf{M}_4$): The phase space $\mathcal{B}$ is a uniform fading memory.

Then, we have the following result.

Lemma 3.6 ([12]) Let ($\mathbf{M}_4$) be satisfied and $z \in BUC(\mathbb{R}, X_0)$ is an asymptotic 1-periodic function. Let $v: \mathbb{R} \rightarrow \mathcal{B}$ be a function defined by $v(t) = z_t$. Then

- (i) v is a $\mathcal{B}$ -valued asymptotic 1-periodic function, and
- (ii) $\sigma(z) = \sigma(v)$.

Lemma 3.7. *Let $f \in BUC(\mathbb{R}, X)$ and let $u \in BUC(\mathbb{R}, X_0)$ is a mild solution to Equation (1.1). Then*

$$\sigma(u) \subset \sigma_\Gamma(P) \cup \sigma(f). \quad (3.3)$$

Proof. By applying (2.1) one gets

$$u_{t+1} = U(t+1, t)u_t + \lim_{n \rightarrow \infty} \int_t^{t+1} U(t+1, s)Y^n f(s) ds. \quad (3.4)$$

For each $n \in \mathbb{N}$, we define

$$[H_n f](t) := \int_t^{t+1} U(t+1, s)Y^n f(s) ds, \quad f \in BUC(\mathbb{R}, X), t \in \mathbb{R}. \quad (3.5)$$

Then, by Proposition 2.6

$$[H_n f](t) \Rightarrow [Hf](t) := \lim_{n \rightarrow \infty} \int_t^{t+1} U(t+1, s)Y^n f(s) ds. \quad (3.6)$$

Taking into account that H_n commutes with S , we have

$$\sigma(H_n f) \subset \sigma(Hf) \subset \sigma(f).$$

Now, for each $\lambda \in \mathbb{C} \setminus \Gamma$ we have

$$[(\lambda - S)v](t) = \lambda u_t - U(t+1, t)u_t + \lim_{n \rightarrow \infty} H_n f = \lambda u_t - P(t)u_t + Hf.$$

Here v is a mapping defined in Lemma 3.6.

Since $(U(t, s))_{t \geq s}$ is 1-periodic strongly continuous evolutionary process, $P(t)$ is 1-periodic and commutes with the translation operator S . Hence,

$$(\lambda - \bar{S})\bar{v} = (\lambda - \bar{P})\bar{v} + H\bar{f}. \quad (3.7)$$

Let $0 \neq \lambda_0 \notin \sigma_\Gamma(P) \cup \sigma(f)$ and let V be a fixed small open neighborhood of λ_0 such that

$$V \cap (\sigma_\Gamma(P) \cup \sigma(f)) = \emptyset.$$

Since (3.7),

$$R(\lambda, \bar{S})\bar{v} = R(\lambda, \mathcal{P})\bar{v} - R(\lambda, \mathcal{P})R(\lambda, \bar{S})\bar{H}\bar{f}.$$

This means $R(\lambda, \bar{S})\bar{v}$ has an analytic extension in V , i.e., $\lambda_0 \notin \sigma(v)$. Thus

$$\sigma(v) \subset \sigma_\Gamma(P) \cup \sigma(f).$$

Finally, by applying Lemma 3.6 one gets $\sigma(u) \subset \sigma_\Gamma(P) \cup \sigma(f)$.

The lemma is proved. [Type equation here.](#)

Theorem 3.8. *Let $(\mathbf{M}_1)$ -($\mathbf{M}_4$) are satisfied and let $\sigma_\Gamma(P) \subset \{1\}$. Assume that u is an asymptotic mild solution of Equation (1.1) such that $u \in BUC(\mathbb{R}, X_0)$ and $f \in BUC(\mathbb{R}, X)$ are asymptotic 1-periodic. Then, u is asymptotic 1-periodic, i.e.,*

$$\lim_{t \rightarrow \infty} (u(t+1) - u(t)) = 0.$$

Proof. Since $f \in BUC(\mathbb{R}, X)$ is asymptotic 1-periodic, $\sigma(f) \subset \{1\}$.

By Lemma 3.7, $\sigma(u) \subset \sigma_\Gamma(P) \cup \sigma(f) \subset \{1\}$.

Combining this with Proposition 3.2 to obtain that u is asymptotic 1-periodic.

4. Conclusion

We have proved a Massera type theorem on the existence of asymptotic periodic solutions to a class of non-densely differential equations with infinite delay arising from bounded continuous ones.

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Acknowledgment. We would like to thank the referee for careful reading of the manuscript. The referee's comments, remarks, corrections, and suggestions have significantly improved the quality of this paper. This research is funded by Hong Duc University under grant number DT-2024-06.