LYAPUNOV–KRASOVSKII FUNCTIONAL APPROACH FOR *H*-STABILITY ANALYSIS OF LINEAR CONTINUOUS-TIME SYSTEMS

Le Thi Oanh[1]

Received: 06 April 2025/ Accepted: 16 August 2025/ Published: 25 August 2025

Https://doi.org/10.70117/hdujs.E10.08.2025.862

Abstract: *In this study, we provide new adequate conditions to demonstrate the h-stability of linear continuous-time systems using the Lyapunov-Krasovskii functional approach, which may be seen as an extension of exponential stability. Additionally, we present a simulation-based example to illustrate the relevance of the obtained conclusions.*

Keywords: *h-stability, Lyapunov-Krasovskii functional, continuous-time systems, linear matrix inequalities.*

1. Introduction

Pinto introduced a new concept of stability, known as *h*-stability, in [1, 2], which is stronger than exponential stability in the study of differential and difference systems. When the origin is an equilibrium point, the goal is to establish stability results for a weak system under specific perturbations. The study of exponential asymptotic stability was extended to include a broader class of systems, known as *h*-systems, by Medina and Pinto in [3].

Analyzing asymptotic stability under non-exponential forms of stability presents significant challenges. Therefore, in the study of differential systems, the concept of *h*-stability is particularly useful and widely applicable. Numerous studies have been conducted on this topic, leading to extensive development (see [4, 5]). However, research on *h*-stability has primarily relied on comparative evaluation methods. To date, no studies have examined *h*-stability for linear continuous-time systems using the Lyapunov–Krasovskii functional function method, which serves as the motivation for the author to conduct this research.

In this paper, we first apply the Lyapunov–Krasovskii functional method to establish a sufficient condition, in the form of a linear matrix inequality, for a continuous linear system to be *h*-stable. This presents a new approach to studying the concept of *h*-stability.

Notation. $R^{n\times m}$ denotes the set of $n\times m$ real matrices. The largest and smallest real parts of the eigenvalues of a matrix P are denoted by $\lambda_{\max}\left(P\right)$ and $\lambda_{\min}\left(P\right)$, respectively.

The transpose of a matrix X is written as X^T, A matrix $M \in R^{n\times m}$ is semi-positive definite, $M > 0$, if $x^T Mx \geq 0$, $\forall x \in R^n$; is positive definite, $M > 0$, if $x^T Mx > 0$, $\forall x \in R^n$ $x \neq 0$.

1 Faculty of Natural Sciences, Hong Duc University; Email: lethioanh@hdu.edu.vn

2. Preliminaries

Consider a class of linear continuous-time systems described by the following equation

$$\dot{x}(t) = Ax(t)\ x(t_0) = x_0, \forall t \geq t_0 \geq 0. \tag{1}$$

Where $x(t) \in R^n$ is the state vector and $A \in R^{n \times m}$ is a given real matrix.

Let the unique solution to system (1) that passes through the initial state $x_0 \in R^n$ at time $t = t_0$ be represented by the equation $x(t) = x(t, t_0, x_0)$.

The definitions of many known types of stability are given below, taken from $[6]$.

Definition 2.1. The system (1) is said to be uniformly stable if for each $\varepsilon > 0$ there is $\delta = \delta(\varepsilon) > 0$, such that

$$\|x(t_0)\| < \delta \Rightarrow \|x(t)\| < \varepsilon, \forall t \geq t_0 \geq 0. \tag{2}$$

Definition 2.2. The system (1) is said to be exponentially stable, if there exist positive constant numbers N and α, such that for each $x_0 \in R^n$ and for any $t_0 \in R^+$

$$\|x(t)\| \leq N.\|x_0\|.e^{-\alpha(t-t_0)},\ \forall t \geq t_0 \geq 0 \tag{3}$$

Definition 2.3. The system (1) is called globally h-stable if there exist $N \geq 1$ and a non-increasing differentiable function $h:[t_0, +\infty) \to [t_0, +\infty)$ such that for each $x_0 \in R^n$, we have

$$\|x(t)\| \leq N.\|x_0\|.h(t)\left[h(t_0)\right]^{-1}, \forall t \geq t_0 \geq 0 \tag{4}$$

where $\left[h(t)\right]^{-1} = \dfrac{1}{h(t)}$.

3. Main results

In this section, we first analyze the *h*-stability analysis of the linear continuous-time system (1) by Lyapunov–Krasovskii functional method. The main result is shown in the following theorem.

Theorem 3.1. *For a given scalar $\beta > 0$, suppose that there is a non-increasing differentiable function $h:[t_0, +\infty) \to [t_0, +\infty)$ subject to $-\left[h(t)\right]^{-1} h(t) < \beta$, and a symmetric positive definite matrix P such that the following linear matrix inequality holds*

$$2\beta P + PA + A^T P < 0 \tag{5}$$

Then, the (1) is globally h-stable.

Proof. We construct a Lyapunov–Krasovskii functional in the form

$$V(t) = x^T(t)\left[h(t)\right]^{-2}\left[h(t)\right]^2 Px(t) \tag{6}$$

First, taking the time-derivative of $V(t)$ along trajectories of system (1) is given by

$$\begin{aligned} \dot{V}(t) &= -2\left[h(t)\right]^{-1} h(t)\left(x^T(t)\left[h(t)\right]^{-2}\left[h(t_0)\right]^2 Px(t)\right) \\ &+2\left[h(t)\right]^{-2}\left[h(t_0)\right]^2 x^T(t).Px(t) \\ &\le 2\beta\left(x^T(t)\left[h(t)\right]^{-2}\left[h(t_0)\right]^2 Px(t)\right) \\ &+2\left[h(t)\right]^{-2}\left[h(t_0)\right]^2 x^T(t).A^T Px(t) \\ &\le \left[h(t)\right]^{-2}\left[h(t_0)\right]^2\left(2\beta x^T(t)Px(t) + 2x^T(t)A^T Px(t)\right) \\ &\le \left[h(t)\right]^{-2}\left[h(t_0)\right]^2 x^T(t)\left(2\beta P + A^T P + PA\right)x(t). \end{aligned} \tag{7}$$

From (5) and (7), we imply that $V(t) \le 0$. Therefore, we obtain

$$V(t) \le V(t_0), \forall t \ge t_0 \ge 0 \tag{8}$$

This leads to

$$x^T(t)\left[h(t)\right]^{-2}\left[h(t_0)\right]^2 Px(t) \le x^T(t_0)Px(t_0) \tag{9}$$

On the other hand

$$x^T(t)\left[h(t)\right]^{-2}\left[h(t_0)\right]^2 Px(t) \ge \lambda_{\min}(P)\left[h(t)\right]^{-2}\left[h(t_0)\right]^2\left\|x(t)\right\|^2 \tag{10}$$

And

$$x^T(t_0)Px(t_0) \le \lambda_{max}(P)\left\|x_0\right\|^2 \tag{11}$$

From (9) - (11), we have

$$\lambda_{\min}(P)\left[h(t)\right]^{-2}\left[h(t)\right]^2\left\|x(t)\right\|^2 \le \lambda_{max}(P)\left\|x_0\right\|^2 \tag{12}$$

Implies

$$\left\|x(t)\right\| \le \sqrt{\frac{\lambda_{max}(P)}{\lambda_{\min}(P)}}\left\|x_0\right\| h(t)\left(h(t_0)\right)^{-1} \tag{13}$$

which shows that system (1) is $h-$ stable. The proof is completed.

Remark 3.1. The concept of global $h-$ stability is a highly flexible definition. The following types of stability can be obtained by selecting different $h-$ functions.

If $h(t) = e\text{-}\alpha t$, then condition (4) becomes

$$\left\|x(t)\right\| < N\left\|x_0\right\| e^{-\alpha t} e^{\alpha t_0} = \sqrt{\frac{\lambda_{max}(P)}{\lambda_{\min}(P)}}\left\|x_0\right\| e^{-\alpha(t-t_0)}, \ \forall t \ge t_0 \ge 0.$$

that is, the system (1) is exponentially stable as in Definition 2.

For given $\lambda > 0$, choose $h(t) = \frac{1}{(1+t)}, t \geq 0$, then condition (4) becomes

$$\|x(t)\| < N\|x_0\| \frac{1}{(1+\lambda)}(\lambda + t_0) < \lambda \sqrt{\frac{\lambda_{max}(P)}{\lambda_{\min}(P)}} \|x_0\| \frac{1}{(1+\lambda)}, \ \forall t \geq t_0 \geq 0,$$

that is, the global polynomial stability of linear continuous-time systems (1) can then be derived. To end this paper, we provide an example to illustrate our result.

Example 3.1. Consider system (1) with the system

$$A = \begin{bmatrix} -4,5 & 0,18 & 0,15 \\ 0,17 & -4,5 & 0,5 \\ 0,01 & 0,02 & -5,2 \end{bmatrix}$$

and $h(t) = e^{-0,95t}$ .It is easy to verify that

$$-[h(t)]^{-1} h(t) = 0,95e^{-0,95t}e^{-0,95t} = 0,95 \leq 0,95 = \beta .$$

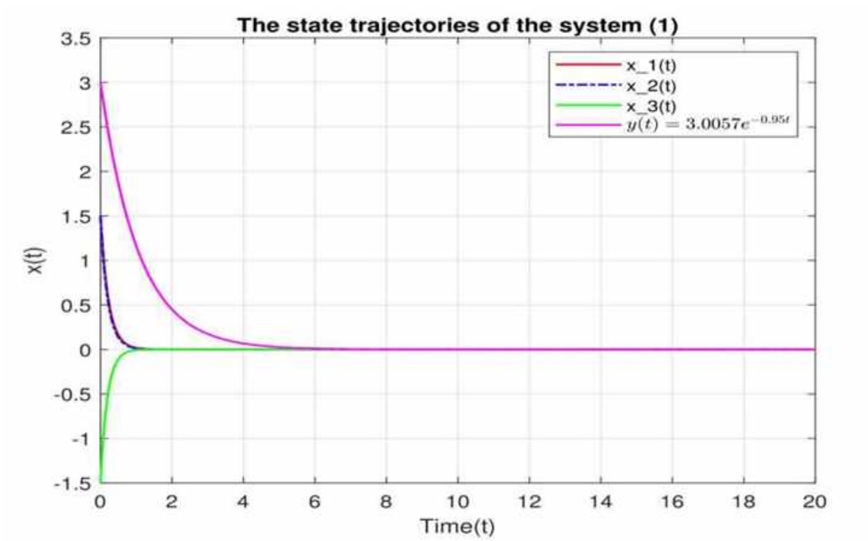

Figure 1: The state trajectories $x_1(t), x_2(t)$, and $x_3(t)$ of the system (1).

According to Theorem 1 and by using the Matlab LMI Toolbox, we find the matrix P as follows

$$P = \begin{bmatrix} 0,2367 & 0,0128 & 0,0063 \\ 0,0128 & 0,2370 & 0,0171 \\ 0,0063 & 0,0171 & 0,1965 \end{bmatrix}$$

In addition, we can calculate $\lambda_{max}\left(P\right)=0,2545$ and $\lambda_{\min}\left(P\right)=0,1902$. We choose initial conditions $x_1\left(0\right)=x_2\left(0\right)=1,5, x_3\left(0\right)=-1,5.$

Then, we have the following estimate

$$\left\|x\left(t\right)\right\|<N\left\|x_0\right\|e^{-\beta t}=\sqrt{\frac{\lambda_{max}\left(P\right)}{\lambda_{\min}\left(P\right)}}\left\|x_0\right\|e^{-\beta t}=3,0057e^{-0,95t}:=y\left(t\right),\ \forall t\geq 0.$$

Hence, according to Theorem 1, the system $\left(1\right)$ is globally *h*-stable, which can be shown in Figure 1.

4. Conclusion

In this work, we use the Lyapunov-Krasovskii functional approach, which may be viewed as an extension of exponential stability, to show the *h*-stability of linear continuous-time systems by providing new sufficient conditions. We also provide a simulation-based example to demonstrate the applicability of the obtained result.

References

[1] M. Pinto (1992), *Stability of nonlinear differential systems*, Appl. Anal. 43, 1-20.

[2] M. Pinto (1984), *Perturbations of asymptotically stable differential systems*, Analysis 4, 161-175.

[3] Medina, R., & Pinto, M. (1996). Stability of nonlinear difference equations. Dynamic Systems and Applications, 2, 397-404.

[4] H. Damak, M. A. Hammami and A. Kicha (2020), *A converse theorem on practical h-stability of nonlinear systems*, Mediterr. J. Math. 17, 1-18

[5] A. B. Makhlouf and M. A. Hammami (2015), *A nonlinear inequality and application to global asymptotic stability of perturbed systems*, Math. Method. Appl. Sci. 38, 2496-2505.

[6] B. B. Hamed, Z. H. Salem and M. A. Hammami (2013), *Stability of nonlinear time-varying perturbed differential equations*, Nonlinear Dyn. 73, 1353-1365.