ZEROS OF FUNCTIONALS ON PARTIAL METRIC SPACES WITH APPLICATION

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Abstract: *We prove a local version of a result obtained recently by Luong anh Hoc on the existence of zeros of functionals on partial metric spaces and apply it to the study of the preservation of zeros of a family of functionals. As a corollary, we derive a preservation result for fixed points of a family of multi-valued mappings in partial metric spaces.*

Keywords: *Partial metric spaces, fixed points, zeros of functionals.*

1. Introduction and preliminaries

In [12], Matthews introduced the concept of a partial metric space, motivated by its potential to model the mathematical semantics of programming languages. A fundamental difference from standard metric spaces is that the distance between a point and itself is not required to be zero. These spaces have been successfully applied in various areas of computer science, including domain theory, programming languages, and semantics ([7,9,11,13,15]). Matthews also established a partial metric analog of the Banach contraction mapping theorem. Following this foundational work, there has been extensive research into the topological properties and fixed-point theory within partial metric spaces ([1-6,8,10,14,16,17]).

We now recall some definitions and basic results in partial metric spaces.

Definition 1.1. [12] Let X be a nonempty subset. A function $p: X \times X \to \mathbb{R}_+$ is said to be a partial metric on X if for any $x, y, z \in X$, the following condition hold:

(p1) $p(x,x) = p(y,y) = p(x,y)$ if and only if $x = y$;

(p2) $p(x,x) \le p(x,y)$;

(p3) $p(x,y) = p(y,x)$;

(p4) $p(x,y) \le p(x,z) + p(z,y) - p(z,z)$.

The pair (X,p) is said to be a partial metric space.

It follows from (p1) and (p2) that if $p(x,y) = 0$, then $x = y$. However, the converse is not true. The condition (p2) means that x minimizes the distance from itself and this distance might be positive. One well-known example of a partial metric space is the pair (X,d) with $X \subset \mathbb{R}_+$ and $p: X \times X \to \mathbb{R}_+$ defined by $p(x,y) = \max\{x,y\}$ for all $x, y \in X$.

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Let (X, p) be a partial metric space. The open p-ball centered at $x \in X$ with radius $r > 0$ is defined by $B_p(x, r) = \{y \in X: p(x, y) < p(x, x) + r\}$. It is well known that the partial metric p generates a T_0 topology τ_p on X with a base being the collection of the open p-balls $\{B_p(x, r): x \in X, r > 0\}$. Let A be a subset of X. The subset A is said to be open (respectively, closed) if it is open (respectively, closed) with respect to the topology τ_p. The set A is said to be bounded if there exist $x_0 \in X$ and $r > 0$ such that $A \subset B(x_0, r)$. We denote by $P(X)$ the set of all nonempty subsets of X, by $C(X)$ the set of all nonempty closed subsets of X and by $CB(X)$ the set of all nonempty closed bounded subsets of X.

Definition 1.2. Let (X, p) be a partial metric space and $\{x_n\}$ be a sequence in X. Then,

(i) $\{x_n\}$ is said to converge to $x \in X$, with respect to τ_p, denoted by $x_n \overset{p}{\to} x$, if $p(x, x) = \lim_{n\to\infty} p(x, x_n)$.

(ii) $\{x_n\}$ is called a Cauchy sequence if $\lim_{m,n\to\infty} p(x_n, x_m)$ exists and is finite. We say that (X, p) is complete if every Cauchy sequence $\{x_n\}$ in X converges, with respect to τ_p, to a point $x \in X$ such that $p(x, x) = \lim_{m,n\to\infty} p(x_n, x_m)$.

(iii) $\{x_n\}$ is said to be 0-Cauchy if $\lim_{m,n\to\infty} p(x_n, x_m) = 0$. We say that (X, p) is 0-complete if every 0-Cauchy sequence in X converges, with respect to τ_p, to a point $x \in X$ such that $p(x, x) = 0$.

Note that if (X, p) is complete, then it is 0-complete. However, as shown in the following example, the converse is not true.

Example 1.1. ([14]). The partial metric space $(\mathbb{Q} \cap \mathbb{R}^+, p)$ with $\mathbb{Q}$ being the set of all rational numbers and $p(x, y) = \max\{x, y\}$ for all $x, y \in \mathbb{Q} \cap \mathbb{R}^+$, is a 0-complete partial metric space but not complete.

Lemma 1.1. ([2]) *Let (X, p) be a partial metric space and $\{x_n\}$ in X be convergent to $x \in X$ with $p(x, x) = 0$. Then, for every $y \in X$, $\lim_{n\to\infty} p(x_n, y) = p(x, y)$.*

Let A, B be subsets of X. The distance from an element $x \in X$ to the set A is defined by $p(x, A) = \inf\{p(x, a): a \in A\}$. The excess of A over B is defined by $e(A, B) = \sup\{d(a, B): a \in A\}$. The generalized Hausdorff distance between A and B is defined by $H(A, B) = \max\{e(A, B), e(B, A)\}$.

Lemma 1.2. ([1]) *Let (X, p) be a partial metric space and A any nonempty set in X. Then $a \in \bar{A} \Leftrightarrow p(a, A) = p(a, a)$, where $\bar{A}$ denotes the closure of A with respect to the partial metric p. Notice that A is closed in (X, p) if and only if $A = \bar{A}$.*

Lemma 1.3. ([4]) *Let (X, p) be a partial metric space, $A, B \in CB(X)$ and $h > 1$. Then, for any $a \in A$, there exists $b \in B$ such that $p(a, b) \le h\, H(A, B)$.*

The following theorem is a special case of a result proven in [10].

Theorem 1.1. [10] *Let (X, p) be a 0-complete partial metric space* and $f: X \to \mathbb{R}_+$ be a function. *Assume that there exists $k \in (0,1), L > 0$ and $\ell > 0$ such that for any $x \in X$, there is some $y \in X$ satisfying the following inequalities: $f(y) \le kf(x)$*

And $p(x,y) - min\{p(x,x), p(y,y)\} \leq L[f(x)]^{\ell}$.

If f *is lower semi-continuous, then there exists* $z \in X$ *such that* $f(x) = 0$.

Our aim of this paper is to give a local version of Theorem 1.1 and apply it to prove a preservation result on the existence of zeros of a family of functionals on partial metric spaces. Consequently, a preservation result for fixed point of a family of multi-valued mappings in partial metric spaces is derived.

2. Main results

Our first result is a local version of Theorem 1.1.

Theorem 2.1. *Let* (X,p) *be* 0*-complete partial metric space,* $x_0 \in X$, $r > 0$ *and* $f: X \to \mathbb{R}_+$ *be a lower semi-continuous function on* X. *Assume that there exists* $k \in (0,1)$, $L > 0$ *and* $\ell > 0$ *such that for any* $x \in B(x_0, r)$, *there is some* $y \in X$ *satisfying the following inequalities:*

$$f(y) \leq kf(x) \tag{1}$$

And

$$p(x,y) - min\{p(x,x), p(y,y)\} \leq L[f(x)]^{\ell} \tag{2}$$

If $[f(x_0)]^{\ell} < (1-k^{\ell})r/L$, *then there exists* $x^* \in B(x_0, r)$ *such that* $f(x^*) = 0$, $p(x^*, x^*) = 0$ *and*

$$p(x_0, x^*) \leq \frac{L[f(x_0)]^{\ell}}{1-k^{\ell}} \tag{3}$$

Proof. We will construct, by induction, a sequence $\{x_n\}$ in X starting from x_0 such that for all $n \geq 0$:

$$x_{n+1} \in B(x_0, r) \tag{4}$$

$$f(x_{n+1}) \leq kf(x_n) \tag{5}$$

$$p(x_n, x_{n+1}) - \min\{p(x_n, x_n), p(x_{n+1}, x_{n+1})\} \leq L[f(x_n)]^{\ell} \tag{6}$$

Indeed, since $x_0 \in B(x_0, r)$, by the assumptions, there exists $x_1 \in X$ such that

$$f(x_1) \leq k\, f(x_0)$$

and

$$p(x_0, x_1) - \min\{p(x_0, x_0), p(x_1, x_1)\} \leq L[f(x_0)]^{\ell}.$$

It follows from the latter inequality that

$$\begin{aligned} p(x_0, x_1) - p(x_0, x_0) \leq\ & p(x_0, x_1) - \min\{p(x_0, x_0), p(x_1, x_1)\} \\ & \leq L[f(x_0)]^{\ell} < (1-k^{\ell})r < r. \end{aligned}$$

Thus, $x_1 \in B(x_0, r)$. Therefore, (4) – (6) hold for $n = 0$.

Suppose for some positive integer m we have generated $x_0, x_1, \dots, x_{m-1}$ satisfying (4) – (6) for $n = 0, 1, \dots, m-1$. Since $x_m \in B(x_0, r)$, by the assumptions, there exists $x_{m+1} \in X$ such that $f(x_{m+1}) \leq kf(x_m)$ and

$$p(x_m, x_{m+1}) - \min\{p(x_m, x_m), p(x_{m+1}, x_{m+1})\} \leq L[f(x_m)]^{\ell}.$$

Thus, (5) and (6) hold for $n = m$. Moreover, since (5) holds for $n = 0,1, \dots, m$, one has

$$f(x_i) \leq kf(x_{i-1}) \leq k^2 f(x_{i-2}) \leq \cdots \leq k^i f(x_0)$$

for all $i = 0,1,\dots,m$. We have

$$\begin{aligned} p(x_0, x_{m+1}) - p(x_0, x_0) &\le \sum_{i=0}^{m} p(x_i, x_{i+1}) - \sum_{i=0}^{m} p(x_i, x_i) \\ &\le \sum_{i=0}^{m} [p(x_i, x_{i+1}) - \min\{p(x_i, x_i), p(x_{i+1}, x_{i+1})\}] \\ &\le \sum_{i=0}^{m} L[f(x_i)]^{\ell} \le \sum_{i=0}^{m} L[k^i f(x_0)]^{\ell} \\ &= L[f(x_0)]^{\ell} \sum_{i=0}^{m} (k^{\ell})^i < \frac{L[f(x_0)]^{\ell}}{1-k^{\ell}} < r\,. \end{aligned}$$

This means that $x_{m+1} \in B(x_0, r)$ and (4) holds for $n = m+1$. Thus, by induction, the construction of the sequence $\{x_n\}$ satisfying (4) - (6) is complete.

By (5), we have for all n that

$$f(x_n) \le k f(x_{n-1}) \le \cdots \le k^n f(x_0) \tag{7}$$

Since $k \in (0,1)$ and $f(x) \ge 0$ for all $x \in X$, by (7) we have

$$\lim_{n\to\infty} f(x_n) = 0.$$

Using (7) and (6), we have for all $m > n \ge 0$ that

$$\begin{aligned} p(x_n, x_m) &\le \sum_{i=n}^{m-1} p(x_i, x_{i+1}) - \sum_{i=n+1}^{m-1} p(x_i, x_i) \\ &\le \sum_{i=n}^{m-1} [p(x_i, x_{i+1}) - \min\{p(x_i, x_i), p(x_{i+1}, x_{i+1})\}] \\ &\le \sum_{i=n}^{m-1} L[f(x_i)]^{\ell} \le \sum_{i=n}^{m-1} L[k^i f(x_0)]^{\ell} \\ &= L[f(x_0)]^{\ell} \sum_{i=n}^{m-1} (k^{\ell})^i \le L[f(x_0)]^{\ell} \sum_{i=n}^{\infty} (k^{\ell})^i \\ &= \frac{L[f(x_0)]^{\ell}}{1-k^{\ell}} (k^{\ell})^n. \end{aligned} \tag{8}$$

Since $(k^{\ell})^n \to 0$ as $n \to \infty$, $\lim_{n,m\to\infty} p(x_n, x_m) = 0$. This implies that $\{x_n\}$ is a 0-Cauchy sequence. Since X is 0-complete, there exists $x^* \in X$ such that $x_n \xrightarrow{p} x^*$ and

$$p(x^*, x^*) = \lim_{n\to\infty} p(x_n, x^*) = \lim_{n\to\infty} p(x_n, x_m) = 0.$$

Since f is lower semicontinuous, we have $f(x^*) \le \liminf_{n\to\infty} f(x_n) = 0$ which implies $f(x^*) = 0$.

In (8), letting $n = 0$, we get $p(x_0, x_m) \leq \frac{L[f(x_0)]^\ell}{1-k^\ell}$ for all $m \geq 1$. Then, by Lemma 1.1, one has

$$p(x_0, x^*) = \lim_{m\to\infty} p(x_0, x_m) \leq \frac{L[f(x_0)]^\ell}{1-k^\ell}.$$

This proves (3). Moreover, the latter inequality implies that $p(x_0, x^*) < r$ i.e., $x^* \in B(x_0, r)$. This ends the proof. ■

We next present a result on the preservation of the existence of zeros for a family of one-parameter functionals on partial metric spaces.

Theorem 2.2. *Let (X, d) be 0-complete partial metric space, Ω be an open subset of X and $\{f_t\}_{t\in[0,1]}$ be a family of lower semi-continuous functionals $f_t: \bar{\Omega} \to \mathbb{R}_+$. Assume that the following conditions hold.*

(i) the set $Q = \{(x,t) \in \Omega \times [0,1]: f_t(x) = 0\}$ is closed in the partial metric space $(X \times [0,1], \rho)$ where $\rho\big((x,t),(y,s)\big) = p(x,y) + |t-s|$ for all $x, y \in X$ and $t, s \in [0,1]$;

(ii) there exist $k \in (0,1)$, $L > 0$ and $\ell > 0$ such that for each $t \in [0,1]$ and for each $x \in \bar{\Omega}$, there is $y \in X$ such that

$$f_t(y) \leq k f_t(x)$$

and

$$p(x,y) - min\{p(x,x), p(y,y)\} \leq L[f_t(x)]^\ell.$$

(iii) there exists a continuous increasing function $\theta: [0,1] \to \mathbb{R}$ such that

$$|f_{t_1}(x) - f_{t_2}(x)| \leq |\theta(t_1) - \theta(t_2)| \text{ for all } t_1, t_2 \in [0,1] \text{ and each } x \in \bar{\Omega}.$$

Then, f_1 has a zero in Ω provided that then f_0 has a zero in Ω.

Proof. Since f_0 has a zero in Ω, Q is a nonempty set. We define the partial order relation $\preccurlyeq$ on Q as follows:

$$(x,t) \preccurlyeq (y,s) \quad \Leftrightarrow \quad t \leq s \quad \text{and} \quad p(x,y) \leq \frac{2L}{1-k^\ell}[\theta(s) - \theta(t)].$$

Let D be a totally ordered subset of Q and set

$$t^* = \sup\{t \in [0,1]: (x,t) \in D\}.$$

Then, there exists a sequence $\{(x_n, t_n)\}$ in D such that $(x_n, t_n) \preccurlyeq (x_{n+1}, t_{n+1})$ and $t_n \to t^*$ as $n \to \infty$. Thus, for all $m > n$, $t_n \leq t_m$ and

$$p(x_n, x_m) \leq \frac{2L}{1-k^\ell}[\theta(t_m) - \theta(t_n)] \tag{9}$$

Since θ is continuous and $t_n \to t^*$ as $n \to \infty$, it follows from (9) that $p(x_n, x_m) \to 0$ as $m, n \to \infty$. Thus, $\{x_n\}$ is a 0-Cauchy in X. By the 0-completeness of X, $\{x_n\}$ converges to some $x^* \in X$. Since Q is closed, $\{(x_n, t_n)\} \subset Q$ and $(x_n, t_n) \to (x^*, t^*)$ as $n \to \infty$, we have $f_{t^*}(x^*) = 0$. We claim that (x^*, t^*) is an upper bound of D. Indeed, letting $m \to \infty$ in (9), one gets

$$p(x_n, x^*) \leq \frac{2L}{1-k^\ell}[\theta(t^*) - \theta(t_n)]$$

for all n. This together with the fact $t_n \le t^*$ for all n implies that $(x_n, t_n) \preccurlyeq (x^*, t^*)$ for all n. Let (x, t) be an any element of D. Then, by the convergence of $\{t_n\}$ to t^*, there exists $N \in \mathbb{N}$ such that $t^* - t_N \le t^* - t$. Hence, $t \le t_N$. This implies that $(x, t) \preccurlyeq (x_N, t_N)$. By the transitivity, we have $(x, t) \preccurlyeq (x^*, t^*)$. Therefore, (x^*, t^*) is an upper bound of D. By the Zorn lemma, Q has a maximal element. Let $(\bar{x}, \bar{t})$ be a maximal element of Q. We claim that $\bar{t} = 1$. Assume to the contrary that $\bar{t} < 1$. By the continuity of θ, we can choose $t \in (\bar{t}, 1)$ and

$$r = \frac{2L}{1 - k^\ell}[\theta(t) - \theta(\bar{t})]^\ell$$

such that $B(\bar{x}, r) \subset \Omega$. By (iii), we have

$$[f_t(\bar{x})]^\ell = |f_t(\bar{x}) - f_{\bar{t}}(\bar{x})|^\ell \le |\theta(t) - \theta(\bar{t})|^\ell = \frac{(1 - k^\ell)r}{2L} < \frac{(1 - k^\ell)r}{L}.$$

By Theorem 2.1, there exists $x \in B(\bar{x}, r)$ such that $f_t(x) = 0$. Thus, $(x, t) \in Q$. This contradicts the maximality of $(\bar{x}, \bar{t})$ in Q. Hence, $\bar{t} = 1$ and $(\bar{x}, 1) \in Q$. That is, f_1 has a zero in Ω. This ends the proof. ■

We finally apply Theorem 2.2 to derive a preservation result for fixed points of a family of multi-valued mappings in partial metric spaces. For some results of this type, we refer the reader to, e.g., [17] and references therein.

Theorem 2.3. *Let (X, p) be a 0-complete partial metric space and $\Omega \subset X$ be an open set. Assume that $\{F_t\}_{t \in [0,1]}$ is a family of multi-valued mappings $F_t: \bar{\Omega} \to CB(X)$ satisfying the following conditions:*

(a) $x \notin F_t(x)$ for all $x \in \bar{\Omega} \setminus \Omega$ and $t \in [0,1]$;

(b) there exist $k \in (0,1), L > 0$ and $\ell > 0$ such that for any $x \in \bar{\Omega}$ there is some $y \in X$ satisfying

$$p(y, F_t(y)) \le kp(x, F_t(x))$$

and

$$p(x, y) - min\{p(x, x), p(y, y)\} \le L[d(x, F_t(x))]^\ell$$

for each $t \in [0,1]$;

(c) there exists an increasing continuous function $\eta: [0,1] \to \mathbb{R}$ such that

$$H_p(F_t(x), F_s(x)) \le |\eta(t) - \eta(s)|, \qquad for\ all\ t, s \in [0,1] and\ for\ each\ x \in \bar{\Omega};$$

(d) for each $t \in [0,1]$, the function $x \mapsto p(x, F_t(x))$ is lower semi-continuous.

Then, F_1 has a fixed point in Ω provided that F_0 has a fixed point in Ω.

Proof. For each $t \in [0,1]$, let $f_t: \bar{\Omega} \to \mathbb{R}_+$ be defined by $f_t(x) = p(x, F_t(x))$ for all $x \in \bar{\Omega}$. Then, by (d), f_t is lower semi-continuous for each $t \in [0,1]$. By (b), f_t satisfies condition (ii) of Theorem 2.2. We next show that $\{f_t\}$ satisfies condition (iii) of Theorem 2.2 with $\theta = h\eta$. Indeed, let $t_1, t_2 \in [0,1]$ and $x \in \bar{\Omega}$. Let y_2 be an arbitrary element in $F_{t_2}(x)$. By Lemma 1.3, there exists $y_1 \in F_{t_1}(x)$ such that $p(y_1, y_2) \le hH(F_{t_1}(x), F_{t_2}(x))$. Then, by (c), we have

$$f_{t_1}(x) = p\left(x, F_{t_1}(x)\right) \leq p(x, y_1) \leq p(x, y_2) + p(y_1, y_2) - p(y_2, y_2)$$
$$\leq p(x, y_2) + hH(F_{t_1}(x), F_{t_2}(x))$$
$$\leq p(x, y_2) + |h\eta(t_1) - h\eta(t_2)|.$$

Since $y_2 \in F_{t_2}(x)$ is arbitrary, it follows from the latter inequality that

$$f_{t_1}(x) \leq f_{t_2}(x) + |h\eta(t_1) - h\eta(t_2)|.$$

Changing the roles of t_1 and t_2, one also gets

$$f_{t_2}(x) \leq f_{t_1}(x) + |h\eta(t_1) - h\eta(t_2)|.$$

Thus,

$$\left|f_{t_1}(x) - f_{t_2}(x)\right| \leq |h\eta(t_1) - h\eta(t_2)|.$$

We now show that the set $Q = \{(x,t) \in \Omega \times [0,1]: f_t(x) = 0\}$ is closed. Let $\{(x_n, t_n)\} \subset Q$ be such that $(x_n, t_n) \to (x^*, t^*) \in \overline{\Omega} \times [0,1]$ as $n \to \infty$. Then, $x_n \to x^*$ and $t_n \to t^*$ as $n \to \infty$. Since $\{f_t\}$ satisfies condition (iii) of Theorem 2.2 with $\theta = h\eta$, one has $0 \leq f_{t^*}(x_n) = \left|f_{t_n}(x_n) - f_{t^*}(x_n)\right| \leq h|\eta(t_n) - \eta(t^*)|$.

It follows from the continuity of η that $\lim_{n\to\infty} f_{t^*}(x_n) = 0$. Then, by the lower semi-continuity of f_{t^*}, $0 \leq f_{t^*}(x^*) \leq \liminf_{n\to\infty} f_{t^*}(x_n) = 0$.

This implies that $f_{t^*}(x^*) = 0$. By (a), (x^*, t^*) must belong to Q. Thus, Q is closed. Since F_t has closed values for all $t \in [0,1]$, x is a fixed point of F_t if and only if x is a zero of f_t. Applying Theorem 2.2, we get the desired conclusion. ■

3. Conclusion

We have proved a local version for a result on the existence of zeros of a functional defined on a partial metric space presented in [10]. Based on this result, we have established a preservation result on the existence of zeros of a family of parametric functionals on partial metric spaces. Consequently, we have derived a preservation result for the existence of fixed points of a family of multi-valued mappings in partial metric spaces. It would be interesting to extend the results of this paper to multi-valued functionals and apply obtained results to fixed point theory. We leave this topic for future works.

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