REACHABLE SET ESTIMATION FOR SINGULAR DISCRETE-TIME NONLINEAR CONTROL SYSTEMS

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Received: 27 February 2024/ Accepted: 15 July 2024/ Published: August 2024

Https://doi.org/10.70117/hdujs.E9.2024.645

Abstract: *The reachable set estimation (RSE) and state-feedback controller design issues for discrete-time singular nonlinear systems with bounded disturbance inputs are the focus of this paper. First, a sufficient condition for estimating the reachable set of the considered system is derived, based on the Lyapunov-Krasovskii functional approach and the singular systems decomposition technique. Under zero initial conditions, that condition ensures the existence of an ellipsoid containing the system state. Second, the RSE is taken into account in the controller design.*

Keywords: *Reachable set estimation, singular systems, control theory.*

1. Introduction

From a theoretical and practical perspective, singular systems represent a significant class of dynamic system models [1,2] because they can describe the dynamic structure of physical systems and involve algebraic relations between nondynamic constraints. Singular systems, also known as differential/difference systems [4, 5], semi-state systems, or descriptor systems [1], are more difficult to study because, in addition to the system's stability, regularity, and the absence of impulses (for continuous singular systems) or causality (for discrete singular systems) must be considered, while standard state-space systems do not address these two issues. Due to the wide range of applications that singular systems have in electrical network analysis, aerospace engineering, economic systems, and physical processes, there has been an increase in interest in singular system research in recent years [9-11]. On the other hand, numerous nonlinear systems are found in the field of practical engineering and processes, such as vehicle positioning, robot positioning, and autopilot systems. In light of this, studying the qualitative behavior of nonlinear systems is crucial for applied models, an area of study that has seen a lot of interest over the past 20 years [6,12].

The set of all possible states that can be attained when the system is started with all permissible initial conditions and inputs is known as the reachable set. In practical engineering, RSE plays a crucial role in guaranteeing safe operation by synthesizing

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controllers to steer clear of undesirable (or unsafe) regions in the state space. If no unsafe states exist in the system's reachable set, the system is considered safe. Therefore, researchers have given the RSE problem a lot of thought and attention over the past few decades [7,8]. For instance, Wang and associates examined the RSE issue for linear systems with time-varying delays and polytopic uncertainties in [7]; RSE was discussed by the writers of [8] for discrete-time Markovian jump systems; RSE of discrete-time singular systems was proposed by Feng et al in [9]. However, as far as we are aware, there haven't been many findings published in the literature up to this point regarding RSE for singular discrete-time nonlinear control systems. This drives the current investigation.

2. Preliminaries

Consider a class of singular nonlinear systems described by the following form

$$Ex(i+1) = Ax(i) + Bu(i) + D\mathrm{w}(i) + f\left(i, x(i), \mathrm{w}(i)\right), \qquad i \in \mathbb{Z}_+ \tag{1}$$

where $x(i) \in \mathbb{R}^n, u(i) \in \mathbb{R}^m$, and $\mathrm{w}(i) \in \mathbb{R}^s$ are state vector, input control, and the disturbance input vector respectively. $E, A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times m}$ and $D \in \mathbb{R}^{n\times s}$ are given real matrices, where E is singular with $rank\left(E\right) = r < n.$ Initial condition of (1) is defined by $x(0) = 0$. In this paper, the disturbance $w(i)$ is satisfying $\mathrm{w}^\top(i)\mathrm{w}(i) \le d^2$. The nonlinear function $f\left(\cdot\right) := f\left(i, x(i), \mathrm{w}(i)\right)$ satisfies condition:

$$f^\top\left(\cdot\right) f\left(\cdot\right) \le a_x^2 x^\top(i) H_x^\top H_x x(i) + a_\mathrm{w}^2 \mathrm{w}^\top(i) H_\mathrm{w}^\top H_\mathrm{w} \mathrm{w}(i), \forall i = 0,1,2,...,+\infty, \tag{2}$$

where the positive scalars a_x, a_w are the bounding parameters, and H_x, H_w are constant matrices.

The reachable set is defined as $\mathrm{R} = \left\{x(i) \in \mathbb{R}^n \mid x(i), \mathrm{w}^\top(i)\mathrm{w}(i) \le d^2,\ x(0) = 0\right\}$. An elipsoid $\varepsilon = \left\{x \in \mathbb{R}^n \mid x^\top Px \le 1, P = P^\top > 0\right\}$ that bounds the reachable set of Equation (1) is formulated. Let us introduce the following definitions.

Definition 1 ([3, 4]). *The pair* $\left(E, A\right)$ *is said to be regular if* $\det\left(sE - A\right)$ *is not identically zero. The pair* $\left(E, A\right)$ *is said to be causal if* $\deg\left(\det\left(sE - A\right)\right) = rank\left(E\right)$. *The singular system in (1) with* $u\left(i\right) = 0$ *is said to be regular and causal, if the pair* $\left(E, A\right)$ *is regular and causal. The singular system in (1) with is* $u\left(i\right) = 0$ *said to be stable if for any scalar* $\epsilon > 0,$ *there exists a scalar* $\delta\left(\epsilon\right) > 0,$ *the solution x(i) of system (1) satisfies* $\delta\left(\epsilon\right) > 0,$ *for any* $i \ge 0$, *moreover* $lim_{i\to\infty}\ x\left(i\right) = 0.$ *The singular system in (1) with* $u\left(i\right) = 0$ *is said to be admissible if it is regular, causal and stable.*

Remark 1. *Note that, if the singular system in (1) with* $u(i)=0$ *is regular and causal, there exist two non-singular matrices* M,N *such that*

$$\overline{E}=MEN=\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}, M\ f(\cdot)=\begin{bmatrix} f_1(\cdot) \\ f_2(\cdot) \end{bmatrix}, \overline{A}=MAN=\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, MD=\begin{bmatrix} D_1 \\ D_2 \end{bmatrix}. \quad (3)$$

Under the state transformation $y(i)=N^{-1}x(i)=\begin{bmatrix} y_1(i) \\ y_2(i) \end{bmatrix}$*, where* $y_1(i)\in\mathbb{R}^r, y_2(i)\in\mathbb{R}^{n-r}$.

Then system in (1) is a restricted system equivalent to the following one

$$y_1(i+1) = A_{11}y_1(i)+A_{12}y_2(i)+D_1w(i)+f_1(.) \quad (4)$$

$$0=A_{21}y_1(i)+A_{22}y_2(i)+D_2w(i)+f_2(.) \quad (5)$$

Using the decomposition (4), we obtain the following auxiliary result.

Lemma 1 ([4]). *System (1) with* $u(i)=0$ *is regular and causal if the matrix* A_{22} *in the decomposition given in (3) is nonsingular.*

Lemma 2 ([9,10]). *Consider system in (1) with* $u(i)=0$ *and* $\mathrm{w}^\top(i)\mathrm{w}(i)\le d^2$. *Let* $V(x(i))$ *be a non-negative functional for discrete-time singular system with* $V(0)=0$, *if there exists a scalar* $\xi\in(0,1)$ *such that*

$$\mathrm{J}(i)=V(x(i+1))-\xi V(x(i))-\frac{1-\xi}{\varpi^2}w^\top(i)w(i)\le 0 \quad (6)$$

then $V(x(i))\le 1$.

Lemma 3 ([13]). *For any matrices* W_1, W_2 *of appropriate dimensions and a symmetric positive definite matrix* Q, *the following inequality holds*

$$-W_1QW_1^\top \le W_2^\top W_1+W_1^\top W_2+W_2^\top Q^{-1}W_2.$$

Assumption 1. $\|A_{22}\|\ge a_x\|M\|\,\|H_xN\|$.

3. Main results

In this section, we will consider the problem of a reachable set of the system (1) with $u(i)=0$. Then, a state feedback controller $u(i)=Kx(i)$ procedure will be proposed such that the resulting closed-loop system is admissible and reachable set of closed-loop system is bounded within the an given ellipsoid.

Theorem 1. *For the given scalar* $0<\alpha<1$. *Assume that there exist symmetric positive matrix* P *and any matrix* S *of appropriate dimension, that the following condition hold:*

$$\Omega<0 \quad (7)$$

where $\Omega = \Omega^T = \left(\Omega_{ij}\right)_{9\times 9}$,

$\Omega_{11} = -\alpha E^\top PE + SL^\top A + A^\top LS^\top,\ \Omega_{12} = SL^\top D,\ \Omega_{13} = SL^\top,\ \Omega_{14} = H_x^\top,\ \Omega_{16} = A^\top P,$

$\Omega_{22} = -\frac{1-\alpha}{d^2} I,\ \Omega_{25} = H_w^\top,\ \Omega_{26} = D^\top P,\ \Omega_{33} = I,\ \Omega_{36} = P,\ \Omega_{44} = \frac{-1}{a_x^2} I,\ \Omega_{55} = -\frac{1}{a_w^2} I,$

$\Omega_{66} = -P,\ \Omega_{15} = \Omega_{23} = \Omega_{24} = \Omega_{34} = \Omega_{35} = \Omega_{45} = \Omega_{46} = \Omega_{56} = 0,$

and $L \in \mathbb{R}^{n\times n}$ *is a constant matrix satisfying* $E^\perp L = 0$ *with* $rank\left(L\right) = n - r$. *Then the system in (1) with* $u\left(i\right) = 0$ *is admissible and the reachable set of system (1) is bounded within the ellipsoid* $\varepsilon\left(\Lambda_\epsilon\right) = \{x \in \mathbb{R}^n \mid x^\top \Lambda_\epsilon x \le 1\}$ *for zero initial condition and disturbance satisfy* $w^\top(i)w(i) \le d^2$, *where*

$$\Lambda_\epsilon = N^{-\top}\begin{bmatrix} \epsilon P & 0 \\ 0 & \frac{1-\epsilon}{\sigma^2} I \end{bmatrix} N^{-1}, \forall 0 < \epsilon < 1, P = \left(M^{-\top} P M^{-1}\right)_{r\times r},$$

$$\sigma = \frac{a_x \| M \| \| H_x N \| + \| A_{21} \|}{\left(\| A_{22} \| - a_x \| M \| \| H_x N \|\right)\sqrt{\lambda_{min}\left(P\right)}} + \frac{a_w \| M \| \| H_w \| d + d \| D_2 \|}{\left(\| A_{22} \| - a_x \| M \| \| H_x N \|\right)},$$

and $M, N, A_{21}, A_{22}, D_2$ *are defined as in (3).*

Proof. We shall begin with showing that system (1) is regular and causal. Since $rank\left(E\right) = r < n$, there exist non-singular matrices M, N satisfiying equations in (3). This implies $N^\top E^\top = \bar{E}^\top M^{-\top}$, we can parameterize $L = \left(E^\top\right)^\perp$ as $L = M^\top \begin{bmatrix} 0 \\ X \end{bmatrix}$. We also denote

$$\bar{S} = N^\top S = \begin{bmatrix} S_{11} \\ S_{21} \end{bmatrix}, \bar{P} = M^{-\top} P M^{-1} = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}. \tag{8}$$

From (7), we have

$$-\alpha E^\top PE + SL^\top A + A^\top LS^\top < 0. \tag{9}$$

By pre- and post-multiplying both sides of (9) with $N^\top$ and N we then obtain

$$-\alpha \bar{E}^\top \bar{P}\, \bar{E} + \bar{S}\begin{bmatrix} 0 & X \end{bmatrix} \bar{A} + \bar{A}^\top \begin{bmatrix} 0 \\ X \end{bmatrix} \bar{S}^T = \begin{bmatrix} \Theta_{11} & \Theta_{12} \\ \Theta_{12}^\top & \Theta_{22} \end{bmatrix} < 0, \tag{10}$$

where

$\Theta_{11} = -\alpha P_{11} + S_{11} X^\top A_{21} + A_{21}^\top X S_{11}^\top, \Theta_{12} = S_{11} X^\top A_{22} + A_{21}^\top X S_{21}^\top,\ \Theta_{22} = S_{21} X^\top A_{22} + A_{22}^\top X S_{21}^\top.$

From (10) it yields $S_{21} X^\top A_{22} + A_{22}^\top X S_{21}^\top < 0$ which implies that A_{22} is nonsingular. Therefore, the pair $\left(E, A\right)$ is regular and causal. Then, the system (1) is regular and causal. Next, we will prove that the ellipsoid provided in Theorem 1 can bound the reachable sets of the singular delay system in (1). We construct the following Lyapunov-Krasovskii functional:

$$V\big(x(i)\big)=x^\top(i)E^\top \mathrm{P}Ex(i) \tag{11}$$

We first define the difference $\Delta V\big(x(i)\big)=V\big(x(i+1)\big)-V\big(x(i)\big)$. Then, along to solution of system (1), we have

$$\begin{aligned}
&\Delta V\big(x(i)\big)-\frac{1-\alpha}{d^2}w^\top(i)w(i)\\
&=x^\top(i+1)E^\top \mathrm{P}Ex(i+1)-x^\top(i)E^\top \mathrm{P}Ex(i)-\frac{1-\alpha}{d^2}w^\top(i)w(i)\\
&=x^T(i+1)E^\top \mathrm{P}Ex(i+1)-\alpha x^\top(i)E^\top \mathrm{P}Ex(i)-\frac{1-\alpha}{d^2}w^\top(i)w(i)+(\alpha-1)V\big(x(i)\big)\\
&=\xi^\top(i)\mathrm{L}^\top \mathrm{P}\,\mathrm{L}\,\xi(i)-\alpha x^\top(i)E^\top \mathrm{P}Ex(i)-\frac{1-\alpha}{d^2}w^\top(i)w(i)+(\alpha-1)V\big(x(i)\big),
\end{aligned} \tag{12}$$

where $\mathrm{L}=\begin{bmatrix}A\ D\ I\end{bmatrix}$ *and* $\xi(i)=\begin{bmatrix}x^\top(i)w^\top(i)\,f^\top(.)\end{bmatrix}^\top$.

We observe that $Ex(i+1)=Ax(i)+f\big(i,x(i),w(i)\big)+Dw(i)=\mathrm{L}\xi(i)$. On the other hand, from inequality (2), we obtain

$$0\le a_x^2\,x^\top(i)H_x^\top H_x x(i)+a_w^2 w^\top(i)H_w^\top H_w w(i)-f^\top(.)f(.). \tag{13}$$

Moreover, from $E^\top L=0$, for any matrix S of appropriate dimension, the following zero-equation holds

$$0=2x^\top(i)\mathrm{S}L^\top Ex(i+1)=2x^\top(i)\mathrm{S}L^\top\big[Ax(i)+Dw(t)+f(.)\big]. \tag{14}$$

Combining (11) to (14) yields

$$V\big(x(i+1)\big)-\alpha V\big(x(i)\big)-\frac{1-\alpha}{d^2}w^\top(i)w(i)\le\xi^\top(i)\big(\Psi+\mathrm{L}^\top \mathrm{P}\,\mathrm{L}\big)\xi(i), \tag{15}$$

where

$$\Psi=\begin{bmatrix}-\alpha E^\top \mathrm{P}E+\mathrm{S}L^\top A+A^\top L\mathrm{S}^\top+a_x^2H_x^\top H_x & \mathrm{S}L^\top D & \mathrm{S}L^\top\\ * & \frac{-1-\alpha}{d^2}I+a_w^2H_w^\top H_w & 0\\ * & * & -I\end{bmatrix}<0.$$

From (6), by Schur complement lemma, we have $\Psi+\mathrm{L}^\top \mathrm{P}\,\mathrm{L}<0$, which together with (15),

$$V\big(x(i+1)\big)-\alpha V\big(x(i)\big)-\frac{1-\alpha}{d^2}w^\top(i)w(i)\le 0.\quad w(i)=0,$$

it yields $\Delta V\left(x(i)\right) \leq V\left(x(i+1)\right)-\alpha V\left(x(i)\right) \leq 0$, which implies the singular system (1) is stable. Besides, by using Lemma 2, from (17) we have $V\left(x(i)\right) \leq 1$. On the other hand, from (12) we can obtain $V\left(x(i)\right)=x^{\top}(i)E^{\top}\mathrm{P}Ex(i)$. Then, $x^{\top}(i)E^{\top}\mathrm{P}Ex(i) \leq 1$. This leads to

$$y^{\top}(i)\left(N^{\top}E^{\top}M^{\top}\right)\left(M^{-\top}\mathrm{P}M^{-1}\right)(MEN)y(i) = y^{\top}(i)\overline{E}^{\top}\overline{\mathrm{P}}\overline{E}y(i) \leq 1$$

That is, $y_1^{\top}(i)\mathrm{P}y_1(i) \leq 1$. Thus, we readily obtain

$$\| y_1(i)\| \leq \frac{1}{\sqrt{\lambda_{min}(\mathrm{P})}} . \tag{16}$$

This proves that $\| y_1(i)\|$ is bounded. Now, we will prove that $\| y_2(i)\|$ is bounded. From (5), we have

$$\begin{aligned}\| A_{22}\| \|y_2(i)\| &= \| A_{21}y_1(i)+D_2w(i)+ f_2(\cdot)\| \leq \| A_{21}\| \| y_1(i)\| + \| D_2\| \| w(i)\| + \|f_2(\cdot)\| \\ &\leq \frac{\| A_{21}\|}{\sqrt{\lambda_{\min}(P)}} + d\| D_2\| + \|f_2(\cdot)\| .\end{aligned} \tag{17}$$

Using condition (2), it is obvious that

$$\begin{aligned}\| f_2(\cdot)\| &\leq \| M\| \sqrt{a_x^2\| H_xN\|^2\| y(i)\|^2 + a_w^2\| H_w\|^2\| w(i)\|^2} \\ &\leq \| M\| \left(a_x\| H_xN\| \| y(i)\| + a_{\mathrm{w}}\| H_w\| \| w(i)\|\right) \\ &\leq a_x\| M\| \| H_xN\| \| y_1(i)\| + a_x\| M\| \| H_xN\| \| y_2(i)\| + a_{\mathrm{w}}\| M\| \| H_w\| d \\ &\leq \frac{a_x\| M\| \| H_xN\|}{\sqrt{\lambda_{\min}(P)}} + a_x\| M\| \| H_xN\| \| y_2(i)\| + a_{\mathrm{w}}\| M\| \| H_w\| d .\end{aligned} \tag{18}$$

Combining (17) with (18), we get

$$\left(\| A_{22}\| - a_x\| M\| \| H_xN\|\right)\| y_2(i)\| \leq \frac{a_x\| M\| \| H_xN\| + \|A_{21}\|}{\sqrt{\lambda_{\min}(\mathrm{P})}} + a_w\| M\| \| H_w\| d + d\| D_2\| . \tag{19}$$

By Assumption 1, we have

$$\| y_2(i)\| \leq \frac{a_x\| M\| \| H_xN\| + \| A_{21}\|}{\left(\| A_{22}\| - a_x\| M\| \| H_xN\|\right)\sqrt{\lambda_{\min}(\mathrm{P})}} + \frac{a_w\| M\| \| H_w\| d + d\| D_2\|}{\left(\| A_{22}\| - a_x\| M\| \| H_xN\|\right)} = \sigma. \tag{20}$$

Consequenly

$$y_2^{\top}\frac{1}{\sigma^2}y_2(i) \leq 1 . \tag{21}$$

Then adding the inequality (16) times ϵ and the inequality in (21) times $1-\epsilon$, we obtain

$$x^{\top}\Lambda_{\epsilon}x = \begin{bmatrix} y_1(i) \\ y_2(i) \end{bmatrix}^{\top} \begin{bmatrix} \epsilon P & 0 \\ 0 & \frac{1-\epsilon}{\sigma^2} I \end{bmatrix} \begin{bmatrix} y_1(i) \\ y_2(i) \end{bmatrix} \leq 1. \tag{22}$$

The above estimation shows that the reachable set of Equation 1 is bounded within the ellipsoid $\varepsilon(\epsilon) = \left\{ x \in \mathbb{R}^n \mid x^{\top}\Lambda_{\epsilon}x \leq 1 \right\}$. The proof is completed.

Now, a state feedback controller for singular nonlinear control system (1) will be designed in the form $u(i) = Kx(i)$. Then, the closed-loop system of (1) can be described as

$$Ex(i+1) = (A+BK)x(i) + Dw(i) + f(i, x(i), \mathrm{w}(i)) \quad , \ i \in \mathbb{Z}_+. \tag{23}$$

Remark 2. On the based decompostion (3), the matrix $A+BK$ can be decomposed as follows: $M(A+BK)N = \begin{bmatrix} A_{c11} & A_{c12} \\ A_{c21} & A_{c22} \end{bmatrix}$. Moreover, the assumption 1 will be as $\| A_{c22} \| \geq a_x \| M \| \| H_x N \|$.

Theorem 2. For given $0 < \alpha < 1$. Assume that there exist a symmetric positive definite matrix P, invertible matrices S, X and any matrix W of appropriate dimensions, such that the following LMI holds

$$\overline{\Omega} < 0 \tag{24}$$

where $\overline{\Omega} = \overline{\Omega}^T = \left(\overline{\Omega}_{ij} \right)_{9\times 9}$,

$\overline{\Omega}_{11} = -\alpha E^{\top}PE + L^{\top}AS^{\top} + L^{\top}BW^{\top} + SA^{\top}L + WB^{\top}L,\ \overline{\Omega}_{12} = L^{\top}D,\ \overline{\Omega}_{13} = L^{\top}S^{\top},\ \overline{\Omega}_{14} = SH_x^{\top},$
$\overline{\Omega}_{16} = SA^{\top} + \mathrm{WB},\ \overline{\Omega}_{22} = \Omega_{22},\ \overline{\Omega}_{25} = \Omega_{25},\ \overline{\Omega}_{26} = D^{\top},\ \overline{\Omega}_{33} = S + S^{\top} + I,\ \overline{\Omega}_{36} = S,\ \overline{\Omega}_{44} = \Omega_{44},$
$\overline{\Omega}_{55} = \Omega_{55},\ \overline{\Omega}_{66} = P - X^{\top} - X,\ \overline{\Omega}_{15} = \overline{\Omega}_{23} = \overline{\Omega}_{24} = \overline{\Omega}_{34} = \overline{\Omega}_{35} = \overline{\Omega}_{45} = \overline{\Omega}_{46} = \overline{\Omega}_{56} = 0.$

Then the singular nonlinear closed-loop system (23) is admissible and the reachable set of system (23) is bounded within the ellipsoid $\varepsilon(\Phi_{\varepsilon})$ for zero initial condition and disturbance satisfy $\mathrm{w}^{\top}(i)\mathrm{w}(i) \leq d^2$. A desired state feedback control gain is obtained as $K = W^{\top}S^{-\top}$, where

$$\Phi_{\epsilon} = N^{-\top} \begin{bmatrix} \epsilon P & 0 \\ 0 & \frac{1-\epsilon}{\gamma^2} I \end{bmatrix} N^{-1}, \forall 0 < \epsilon < 1, P = \left(M^{-\top}P^{-1}M^{-1} \right)_{r\times r},$$

$$\gamma = \frac{a_x \| M \| \| H_x N \| + \| A_{c21} \|}{\left(\| A_{c22} \| - a_x \| M \| \| H_x N \| \right)\sqrt{\lambda_{\min}(P)}} + \frac{a_{\mathrm{w}} \| M \| \| H_{\mathrm{w}} \| d + d \| D_2 \|}{\left(\| A_{c22} \| - a_x \| M \| \| H_x N \| \right)},$$

and $L = \left(E^{\top} \right)^{\perp}$ *with* $rank(L) = n - r$ *and other notations are defined as in Theorem 1.*

Proof. According to Theorem 1 with A is replaced by $A+BK$ matrix, we let S as an invertible matrix. By pre- and post-multiplying both sides of (7) with $G = diag\left\{ \mathrm{S}^{-1}, I,\ \mathrm{S}^{-1},\ I,\ I,\ \mathrm{P}^{-1} \right\}$ and its transpose, respectively. On the other hand, based on the singular value decomposition, there exist orthogonal matrices Z,Y such that the matrix E can be represented as $E = Z\begin{bmatrix} J_r & 0 \\ 0 & 0 \end{bmatrix}Y^{\top}$, where $Z^{\top}Z = Y^{\top}Y = I_n$ and J_r is an invertible matrix. We now define $\mathrm{S}^{-\top} = Y\begin{bmatrix} \mathrm{S}_{11} & 0 \\ \mathrm{S}_{21} & \mathrm{S}_{22} \end{bmatrix}Y^{\top}$, where $\mathrm{S}_{11}, \mathrm{S}_{22}$ and S_{21} are non-singular matrices. Then, we have

$$E\mathrm{S}^{-\top} = Z\begin{bmatrix} J_r\mathrm{S}_{11}J_r^{-1} & 0 \\ 0 & \mathrm{S}_{22} \end{bmatrix}Z^{\top}Z\begin{bmatrix} J_r & 0 \\ 0 & 0 \end{bmatrix}Y^{\top}. \tag{25}$$

Let $X = Z\begin{bmatrix} J_r\mathrm{S}_{11}J_r^{-1} & 0 \\ 0 & \mathrm{S}_{22} \end{bmatrix}Z^{\top}$. It follows from (25) that $E\mathrm{S}^{-\top} = XE$, by which we readily obtain

$$-\alpha\mathrm{S}^{-\top}E^{\top}PE\mathrm{S}^{-\top} = -\alpha E^{\top}X^{\top}PXE \tag{26}$$

In addition, by utilizing the matrix inequality given in Lemma 3, we have

$$-P^{-1} \le XPX - X^{T} - X, \qquad -S^{-1}S^{-\top} \le S^{-1} + S^{-\top} + I. \tag{27}$$

Now, we let $S = \mathrm{S}^{-\top}, P = X^{\top}\mathrm{P}X$ and $W = \mathrm{S}^{-1}K^{\top}$, we get the LMI condition (24). Morever, the controller gain can be obtained as $K = W^{\top}S^{-\top}$ and the reachable set of closed-loop (23) is contained in the ellipsoid $\varepsilon(\Phi_{\epsilon})$. The proof is completed.

5. Conclusion

The problems of RSE and state-feedback controller design have been examined in this paper. A new condition for the solvability of the investigated problems have been presented in terms of matrix inequality. The usefulness of the suggested techniques has been illustrated with numerical examples and simulation results. It should be noted that this paper's developed methods are still somewhat conservative. So, one goal for the future is to create better methods to enhance the current outcomes even more.

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