A RANDOM FIXED POINT THEOREM FOR COMPLETELY RANDOM OPERATORS

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Abstract: *In this paper, we present a random fixed point theorem for completely random operators satisfying a rational contractivity condition. Examples are also given to illustrate our results.*

Keywords: *Randon fixed point, completely random operator, probability space.*

1. Introduction

Fixed point theory is one of the most powerful tools inmathematics. It has many applications in many branches of mathematics and other sciences. Fixed point methods have been proven particularly helpful in the study of theories of differential equations, integral equations, and functional integral equations. They have also been proven beneficial in optimization theory and a variety of other fields, including biology, chemistry, economics, engineering, game theory, and physics.

Starting from Brouwer's fixed point theorem (1910), Banach's contraction principle (1922) and Schauder's fixed point theorem (1930), the fixed point theory has been developed in many directions. In the mid 1950s, O. Hans and A. Spacek initiated to prove fixed point theorems for random operators in separable metric spaces (see, [1, 2]). These results are stochastic generalizations of Banach's fixed point theorem. In 1966, A. Mukherjee [3] generalized Schaduer's fixed point theorem and presented a random fixed point theorem in atomic probability measure spaces. Specially, in 1976, A. T. Bharucha-Reid published an interesting survey article on fixed point theorems for random operators [4]. Since then many authors have generalized existing deterministic and random fixed point theorems to obtain new random fixed poin theorems (see, e.g., [5, 6, 7, 8, 9] and references therein).

In this paper, following the idea and techniques by D. H. Thang and P. T. Anh in [7, 8], we prove a fixed point theorem for completely random operators satisfying a new condition involving a rational expression. We also present two examples to illustrate the obtained result.

2. Preliminaries

In this section, we recall some definitions and basic results concerning completely random operators.

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Let X be a metric space and $\mathcal{B}(X)$ be the Borel σ-algebra of X (the smallest σ-algebra containing all open subsets of X). Let $(\Omega, \mathcal{F})$ be a measurable space. A mapping $\xi: \Omega \to X$ is called $\mathcal{F}$-measurable if

$$\xi^{-1}(B) = \{\omega \in \Omega: \xi(\omega) \in B\} \in \mathcal{F}$$

for all $B \in \mathcal{B}(X)$. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. If $\xi: \Omega \to X$ is $\mathcal{F}$-measurable then ξ is called a X-valued random variable. We denote by $L_0^X(\Omega)$ the set of all equivalent class of X-valued random variables. This set is equipped with the topology of convergence in probability, namely, the basis neighborhoods for this topology are of the form

$$V(u_0, \varepsilon, \alpha) = \{u \in L_0^X(\Omega): \mathbb{P}\{\|u - u_0\| > \varepsilon\} < \alpha\}.$$

Note that this topology is metrizable. The metric d on $L_0^X(\Omega)$ that induces this topology can be given by

$$d(u, v) = \mathbb{E}\frac{\| u - v \|}{1 + \| u - v \|}.$$

Under this metric, $L_0^X(\Omega)$ is a complete metric space (see [7]) and a sequence $(\xi_n) \subset L_0^X(\Omega)$ converges to ξ if and only if (ξ_n) converges to ξ in probability.

Definition 2.1. [10] Let X, Y be two separable Banach spaces.

(i) A mapping $F: \Omega \times X \to Y$ is said to be a random operator if for each fixed x in X, the mapping $\omega \mapsto F(\omega, x)$ is measurable.

(ii) A random operator $F: \Omega \times X \to Y$ is said to be continuous if for each ω in Ω the mapping $x \mapsto F(\omega, x)$ is continuous.

Definition 2.2. [7] Let X, Y be two separable Banach spaces.

(i) A mapping $\Phi: \mathrm{L}_0^X(\Omega) \to L_0^Y(\Omega)$is called a completely random operator.

(ii) The completely random operator Φ is said to be continuous in probability if the mapping $\Phi: \mathrm{L}_0^X(\Omega) \to L_0^Y(\Omega)$ is continuous, i.e., for each sequence (u_n) in $L_0^X(\Omega)$ such that $\lim_n u_n = u$ in probability, we have $\lim_n \Phi u_n = \Phi u$ in probability.

(iii) The completely random operator Φ is said to be an extension of a random operator $F: \Omega \times X \to Y$ if for each x in X

$$\Phi x(\omega) = F(\omega, x) \ a.s.,$$

where for each x in X, x denotes the random variable u in $L_0^X(\Omega)$ given by $u(\omega) = x$ a.s.

Definition 2.3. Let $\Phi: L_0^X(\Omega) \to L_0^X(\Omega)$ be a completely random operator. An X-valued random variable $\xi \in L_0^X(\Omega)$ is called a fixed point of Φ if $\Phi\xi = \xi$ a.s.

3. Main results

From now on, we alway assume that $(\Omega, \mathcal{F}, \mathbb{P})$ is a complete probability space and X is a separable Banach space. Our main result is stated as follows.

Theorem 3.1. Let $\Phi: L_0^X(\Omega) \to L_0^X(\Omega)$ be a continuous in probability completely random operator and α, L, ℓ be positive real numbers with $\ell \leq 1/\alpha$. Assume that for any random variables $u, v \in L_0^X(\Omega)$ and for any $0 < t < \ell$, we have

$$\mathbb{P}(\| \Phi u - \Phi v \| > t) \leq \mathbb{P}\left(\frac{\| u - v \|}{1 + \alpha \| u - v \|} + L \| u - \Phi v \| > t\right). \tag{3.1}$$

If there exist $p > 0$ and $u_0 \in L_0^X(\Omega)$ such that $\mathbb{E}\|\Phi u_0 - u_0\|^p < \infty$, then Φ has a random fixed point in $L_0^X(\Omega)$.

Proof. Set $k(t) = 1 - \alpha t$ for all $t \in (0, \ell)$. Then, k is deceasing in $(0, \ell)$ and $k(t) \in (0,1)$ for all $t \in (0, \ell)$. We have

$$\frac{1}{k\left(\frac{t}{k(t)}\right)} > \frac{1}{k(t)} \text{ for all } t \in \left(0, \frac{\ell}{2}\right).$$

Thus, for each $u, v \in L_0^X(\Omega)$ and for all $s \in (0, \ell), t \in (0, \ell/2)$, one has

$$\left\{\| u - v \| > \frac{s}{k\left(\frac{t}{k(t)}\right)}\right\} \subset \left\{\| u - v \| > \frac{s}{k(t)}\right\}$$

and, therefore,

$$\mathbb{P}\left(\| u - v \| > \frac{s}{k\left(\frac{t}{k(t)}\right)}\right) \leq \mathbb{P}\left(\| u - v \| > \frac{s}{k(t)}\right). \tag{3.2}$$

Let $u_0 \in L_0^X(\Omega)$ be such that $\mathbb{E}\|\Phi u_0 - u_0\|^p < \infty$. We construct the sequence $\{u_n\}$ in $L_0^X(\Omega)$ defined by

$$u_{n+1} = \Phi u_n,\ n = 0, 1, 2, \cdots$$

We are going to show that $\{u_n\}$ is a Cauchy sequence in $L_0^X(\Omega)$. For each n, we have

$$\begin{aligned}
\mathbb{P}(\|u_{n+1} - u_n\| > t) &= \mathbb{P}(\|\Phi u_n - \Phi u_{n-1}\| > t) \\
&\leq \mathbb{P}\left(\frac{\|u_n - u_{n-1}\|}{1 + \alpha\|u_n - u_{n-1}\|} + L\|u_n - \Phi u_{n-1}\| > \frac{t}{k(t)}\right) \\
&= \mathbb{P}\left(\frac{\|u_n - u_{n-1}\|}{1 + \alpha\|u_n - u_{n-1}\|} > t\right) \\
&= \mathbb{P}\left(\|u_n - u_{n-1}\| > \frac{t}{1 - \alpha t}\right).
\end{aligned}$$

Using (3.2), one has

$$\begin{aligned}
\mathbb{P}(\|u_{n+1} - u_n\| > t) &\leq \mathbb{P}\left(\|u_n - u_{n-1}\| > \frac{t}{k(t)}\right) \\
&\leq \mathbb{P}\left(\|u_{n-1} - u_{n-2}\| > \frac{t}{k(t)k\left(\frac{t}{k(t)}\right)}\right) \\
&\leq \mathbb{P}\left(\|u_{n-1} - u_{n-2}\| > \frac{t}{[k(t)]^2}\right) \\
&\leq \cdots \leq \mathbb{P}\left(\|u_1 - u_0\| > \frac{t}{[k(t)]^n}\right) \\
&= \mathbb{P}\left(\|\Phi u_0 - u_0\| > \frac{t}{k^n}\right)
\end{aligned}$$

where $k = k(t)$. By Chebyshev's inequality, we have

$$\mathbb{P}(\|u_{n+1} - u_n\| > t) \leq \mathbb{P}\left(\|\Phi u_0 - u_0\| > \frac{t}{k^n}\right) \leq \mathbb{E}\|\Phi u_0 - u_0\|^p \frac{(k^n)^p}{t^p} = C\frac{(k^n)^p}{t^p},$$

where $C = \mathbb{E}\|\Phi u_0 - u_0\|^p$.

For $k < x < 1$, set $q = \frac{x}{k}$. We have $r > 1$ and

$$(q-1)\left(\frac{1}{q} + \frac{1}{q^2} + \cdots + \frac{1}{q^m}\right) + \frac{1}{q^m} = 1 \text{ for all } m \geq 1$$

Thus, for all $t \in (0,1/\alpha)$ and for all $m, n \in \mathbb{N}$, one has

$$\begin{aligned}
\mathbb{P}(\|u_{n+m} - u_n\| > t) &\leq \mathbb{P}(\|u_{n+m} - u_n\| > (1 - 1/q^m)t) \\
&\leq \mathbb{P}(\|u_{n+m} - u_{n+m-1}\| > t(q-1)/q^m) \\
&\quad + \cdots + \mathbb{P}(\|u_{n+1} - u_n\| > t(q-1)/q) \\
&= \frac{C}{[(q-1)t]^p}[(q^m)^p(k^{n+m-1})^p + \cdots + q^p(k^n)^p] \\
&= \frac{C}{[(q-1)t]^p}(k^n)^p q^p\left[(kq)^{p(m-1)} + \cdots + (kq)^p + 1\right] \\
&= \frac{C}{[(q-1)t]^p}(k^n)^p q^p \frac{1 - (kq)^m p}{1 - (kq)^p} \\
&\leq \frac{Cq^p}{[(q-1)t]^p[1 - (qr)^p]^n} k^{np}
\end{aligned}$$

Since

$$\lim_{n\to\infty} \frac{Cq^p}{[(q-1)t]^p[1 - (qr)^p]} k^{np} = 0,$$

we have $\lim_{n\to\infty} \mathbb{P}(\|u_{n+m} - u_n\| > t) = 0$

for all $t \in (0, \ell/2)$. This implies that $\{u_n\}$ is a Cauchy sequence in $L_0^X(\Omega)$. Thus, there exists $\zeta \in L_0^X(\Omega)$ such that $p - \lim_{n\to\infty} u_n = \zeta$. It follows from $u_{n+1} = \Phi u_n$ and the continuity in probability of Φ that $\zeta = \Phi\zeta$. That is, ζ is a random fixed point of Φ. This ends the proof.

Remark 3.1. It is worth remarking that a fixed point of operators in theorems presented in [7,8] is unique while fixed points of operators in our theorem are not necessarily unique.

The following simple example showing that an operator satisfying conditions in Theorem 3.1 may have many random fixed points.

Example 3.1. Consider the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where $\Omega = [0,1], \mathcal{F}$ is the σ-algebra of Lebesgue measurable subsets of $[0,1]$ and $\mathbb{P}$ is the Lebesgue mesuare on Ω. Let

$X = \mathbb{R}$ and $\Phi: L_0^X(\Omega) \to L_0^X(\Omega)$ be defined by $\Phi u(\omega) = u(\omega)$ for all $\omega \in [0,1]$ and for all $u \in L_0^X(\Omega)$.

Let $\alpha = 1$ and $L = 1$. For $t > 0$, set

$$A = \{\omega \in \Omega: \| \Phi u(\omega) - \Phi v(\omega) \| > t\} = \{\omega \in \Omega: \| u(\omega) - v(\omega) \| > t\}$$

and

$$\begin{aligned}
B &= \left\{\omega \in \Omega: \frac{\| u(\omega) - v(\omega) \|}{1 + \alpha \| u(\omega) - v(\omega) \|} + L \| u(\omega) - \Phi v(\omega) \| > t\right\} \\
&= \left\{\omega \in \Omega: \frac{\| u(\omega) - v(\omega) \|}{1 + \| u(\omega) - v(\omega) \|} + \| u(\omega) - v(\omega) \| > t\right\}.
\end{aligned}$$

It is evident that $A \subset B$ for each $t > 0$. It follows that $\mathbb{P}(A) \leq \mathbb{P}(B)$. Hence, the inequality (3.1) holds. Therefore, all conditions in Theorem 3.1 are satisfied. By Theorem 3.1, Φ has a random fixed point in $L_0^X(\Omega)$. In fact, each $u \in Ł_0^X(\Omega)$ is a random fixed point of Φ. It means that Φ has infinitely many fixed points.

Corrolary 3.1. Let $\Phi: L_0^X(\Omega) \to L_0^X(\Omega)$ be a continuous in probability completely random operator and α, ℓ be positive real numbers with $\ell \leq 1/\alpha$. Assume that for any random variables $u, v \in L_0^X(\Omega)$ and for any $0 < t < \ell$, we have

$$\mathbb{P}(\| \Phi u - \Phi v \| > t) \leq \mathbb{P}\left(\frac{\| u - v \|}{1 + \alpha \| u - v \|} > t\right).$$

If there exist $p > 0$ and $u_0 \in L_0^X(\Omega)$ such that $\mathbb{E}\|\Phi u_0 - u_0\|^p < \infty$, then Φ has a unique random fixed point in $L_0^X(\Omega)$.

Proof. The existence of a random fixed point ζ for Φ follows Theorem 3.1. We now prove the uniqueness of ζ. Assume that η is another fixed point of Φ such that $\eta \neq \zeta$. Them, $\mathbb{P}(\| \zeta - \eta \| > t) > 0$ for some $t > 0$. We may assume that $t \in (0, \ell/2)$. By (3.3) and arguing as in the proof of Theorem 3.1, we have for any positive integer n that

$$\begin{aligned}
\mathbb{P}(\| \zeta - \eta \| > t) &= \mathbb{P}(\| \Phi\zeta - \Phi\eta \| > t) \\
&\leq \mathbb{P}\left(\frac{\| \zeta - \eta \|}{1 + \alpha \| \zeta - \eta \|} > t\right) \\
&= \mathbb{P}\left(\| \zeta - \eta \| > \frac{t}{k(t)}\right) \\
&\leq \cdots \\
&\leq \mathbb{P}\left(\| \zeta - \eta \| > \frac{t}{[k(t)]^n}\right)
\end{aligned}$$

where $k(t) = 1 - \alpha t \in (0,1)$. Letting $n \to \infty$ in the latter in equality, we get $\mathbb{P}(\| \zeta - \eta \| > t) \leq 0$. This is a contradiction. Therefore, Φ has a unique random fixed point.

We next present an example to support the latter result.

Example 3.2. We consider the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where $\Omega = [0,1], \mathcal{F}$ is the σ-algebra of Lebesgue measurable subsets of $[0,1]$ and $\mathbb{P}$ is the Lebesgue mesuare on Ω. Let $X = \mathbb{R}$ and $\Phi: L_0^X(\Omega) \to L_0^X(\Omega)$ be defined by

$$\Phi u(\omega) = \begin{cases} \frac{1}{2} u(2\omega) & \text{if } 0 \leq \omega \leq \frac{1}{2} \\ \frac{1}{3} u(2\omega - 1) & \text{if } \frac{1}{2} < \omega \leq 1. \end{cases}$$

Set

$$\begin{aligned}
A = \quad & \{\omega \in \Omega: \| \Phi u(\omega) - \Phi v(\omega) \| > t\} \\
= \quad & \left\{\omega \in \left[0, \frac{1}{2}\right] : \| \Phi u(\omega) - \Phi v(\omega) \| > t\right\} \\
& \cup \left\{\omega \in \left[\frac{1}{2}, 1\right] : \| \Phi u(\omega) - \Phi v(\omega) \| > t\right\} \\
= \quad & \left\{\omega \in \left[0, \frac{1}{2}\right] : \| u(2\omega) - v(2\omega) \| > 2t\right\} \\
=: \quad & A_1 \cup A_2.
\end{aligned}$$

and

$$B = \left\{\omega \in \Omega: \frac{\| u(\omega) - v(\omega) \|}{1 + \| u(\omega) - v(\omega) \|} > t\right\}$$
$$= \left\{\omega \in \Omega: \| u(\omega) - v(\omega) \| > \frac{t}{1-t}\right\}.$$

For $0 < t < \ell := \frac{1}{2}$, we have $\frac{t}{1-t} < 2t$. Thus,

$$C := \{\omega \in \Omega: \| u(\omega) - v(\omega) \| > 2t\} \subset \left\{\omega \in \Omega: \| u(\omega) - v(\omega) \| > \frac{t}{1-t}\right\},$$

and $\mathbb{P}(C) \leq \mathbb{P}(B)$. On the other hand, we can easily see that $C = 2A_1$ and hence $\mathbb{P}(C) = 2\mathbb{P}(A_1)$.

Since $\frac{t}{1-t} < 3t$ for all $0 < t < \ell$, we have

$$D := \{\omega \in \Omega: \| u(\omega) - v(\omega) \| > 3t\} \subset \left\{\omega \in \Omega: \| u(\omega) - v(\omega) \| > \frac{t}{1-t}\right\},$$

and $\mathbb{P}(\boldsymbol{D}) \leq \mathbb{P}(\boldsymbol{B})$. Moreover,

$$A_2 = \left\{\omega \in \left[0, \frac{1}{2}\right] : \| u(2\omega) - v(2\omega) \| > 3t\right\}$$

and we also see that $D = 3A_2$. Hence, $\mathbb{P}(D) = 2\mathbb{P}(A_2)$.

We have

$$\mathbb{P}(A) = \mathbb{P}(A_1) + \mathbb{P}(A_2) = \frac{1}{2}\mathbb{P}(C) + \frac{1}{2}\mathbb{P}(D) \leq \frac{1}{2}\mathbb{P}(B) + \frac{1}{2}\mathbb{P}(B) = \mathbb{P}(B).$$

Therefore, (3.3) holds for $\alpha = 1$ and $\ell = 1/2$. One can see that $u = 0$ is the unique random fixed point of Φ.

4. Conclusion

We have proved a new random fixed point for completely random operators and presented two examples to suport the obtained results. It is interesting to continue investigating new fixed point results for completely random operators satisfying new contractive conditions.

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