THE INVARIANT PROPERTIES ON THE CS-MODULES AND APPLICATIONS

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Received: 30 October 2023/ Accepted: 15 July 2024/ Published: August 2024

Https://doi.org/10.70117/hdujs.E9.2024.630

Abstract: *In this paper, we prove some invariant properties on the CS-module and application. We also prove some preserving correspondences of closed submodules, and then apply it to transfer the CS, max CS, min CS, max-min CS and (strongly) FI-extending properties of a module to its endomorphism ring. Endomorphism rings of modules with Baer and Rickart properties are studied. Furthermore, the right-left symmetry of Goldie, CS and max-min CS conditions on the endomorphism rings of prime and semiprime modules are investigated. Some examples are discussed to guarantee that our main results make sense.*

Keywords: *Baer modules, CS modules, FI-extending modules, Goldie modules, max-min CS modules, prime modules, semiprime modules, right Rickart property.*

1. Introduction

Throughout this paper, R is an associative ring with identity, and M is a unitary right $R-$module with its endomorphism ring $S =: End(M_R)$. We denote $X \leq M$ (resp. $X \leq^* M$ for a submodule (resp. an essential submodule) X of M. A submodule $X \leq M$ is a *closed submodule* if $X \leq^* Y$ implies $X = Y$ for any submodule Y of M. For every $X \leq M$ there exists a maximal essential extension Y containing X. Then Y is a closed submodule and called a *closure* of X. We write $r_X(Y)$ and $l_X(Y)$ for the right annihilator and the left annihilator of Y in X, respectively. We denote the uniform dimension of the module M_R by $u - dim(M_R)$.

We adopt the notions of primeness and semi-primeness in module category introduced by N.V. Sanh et al. [11]. An X-annihilator X of M is a submodule for some subset T of S. A submodule X of M is *fully invariant* if $f(X) \leq X$ for every endomorphism f. For a submodule X of M we write $I_X = \{ f \in S \mid f(M) \leq X \} = Hom(M, X)$. For a subset K of S, we write $KM = K(M) := \sum_{f \in K} f(M)$. It is clear that I_X is a right ideal of S and KM is a submodule of M. M is *retractable* if $I_X \neq 0$ for every nonzero submodule X. M is a *self-generator* if it generates every submodule, i.e. $X = I_X(M)$ for every submodule X. Obviously, every self-generator is retractable. M is *nonsingular* if for any $m \in M$, $r_R(m) \leq^* R_R$ implies $m=0$.

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Recall that a *CS module* (resp. *min CS module*) provides that every (resp. minimal or uniform) closed submodule is a direct summand. *M* is called a *max CS module* if every maximal closed submodule with nonzero left annihilator in *S* is a direct summand. *M* is a *max-min CS module* if it is both max *CS* and min *CS*. A ring *R* is a right CS (resp. right max CS or right min CS or right max-min CS) ring if R_R is a CS (resp. max CS or min CS or max-min CS) module. Analogous notions on the left side of ${}_R R$ are defined similarly. We refer readers to [6] and [13] for more details about max CS, min CS, max-min CS rings and modules.

According to G.F. Birkenmeier et al. [4], *M* is a (*strongly) FI-extending module* if every fully invariant submodule of *M* is essential in a (fully invariant) direct summand of *M*. *R* is a right (strongly) *FI*-extending if R_R is a (strongly) *FI*-extending module.

Symmetry of extending properties in associative rings are extensively investigated in many papers [2, 3, 4, 6]. Our work aims to generalize such the investigations to modules as well as some results in [8, 10]. In this paper, the extending properties include the *CS*, *max CS*, *min CS*, *max-min CS and (strongly) FI*-extending ones. An extending module means a module having an extending property as mentioned, not just *CS*. We refer readers to a beautiful monograph written by Tercan A. and Yücel C.C.[15] that comprehensively presents extending properties, related concepts and generalizations. We introduce the condition (III) on modules (in which each submodule has a unique closure) to prove a correspondence theorem between closed submodules of a module *M* and closed right ideals of its endomorphism ring *S*. The extending properties are mutually transferred from *M* to *S* and vice versa. In addition, we prove a theorem in which *M* and *S* share Baer and Rickart properties.

The right-left symmetry of the Goldie and *CS* conditions in the endomorphism ring *S* is comprehensively investigated with more general results in comparison with those of D.V. Thuat et al. [12, 13]. Some examples are discussed in the end of this paper to draw a clear picture about the meaning of our work.

2. On the fully invariant submodules with CS conditions

In this part, we introduce some properties of fully invarian submodules and apply to some class of modules.

The first one, we identify three conditions as follows:

(C_1) Every submodule is essential in a direct summand.

It is equivalent to say that every closed submodule is a direct summand.

(C_2)Every submodule that is isomorphic to a direct summmand is itself a direct summand.

(C_3)Direct sum of two direct summands with zero intersection is a direct summand.

Right *R*-module *M* is called *CS* module it satifies (C_1) condition.

A module is called continuous if it satifies both the (C_1) and (C_2) conditions.

A module is called quasi- continuous if it satifies both the (C_1) and (C_3) conditions.

A ring R is called CS ring (respectively continuous, quasi-continuous) if the module R_R has the corresponding property.

Let A,B be submodules of M. We say that B is a complement of A on which B is maximal with respect to $A \cap B = 0$. We have $N \subseteq^* M$ if and only if 0 is a complement of N.

It is easy to see that an indecomposable module is a (C_3) module.

M is a CS module if and only if any it's direct summand is a uniform module (that is $X \cap Y = 0$ for all nonzero submodules X,Y of M.

M is (C_2) module if and only if monomorphisms in *end(M)* are isomorphisms.

$\mathbb{Z}-$ module $\mathbb{Z}_2$ and $\mathbb{Z}_8$ are having (C_1), (C_2), (C_3) conditions, but their direct sum $N = \mathbb{Z}_2 \oplus \mathbb{Z}_8$ is not a (C_1) module.

In fact, let $S = \mathbb{Z}_2 \oplus 0$, $K = \mathbb{Z}(1 + 2\mathbb{Z}, 2 + 8\mathbb{Z})$. We can see that K is contained in only two direct summands $\mathbb{Z}_2 \oplus 0$ and K = $\mathbb{Z}$ (1+2 $\mathbb{Z}$,2+8 $\mathbb{Z}$) and is essential in neither. Moreover N is not (C_2) because the $0 \oplus \mathbb{Z}$ (4 +8 $\mathbb{Z}$) is not a summand but isomorphic to the summand $\mathbb{Z}_2 \oplus 0$. Hence a direct sum of modules satifying (C_1), (C_2) conditions are not inherit the same property.

We have known that module $\mathbb{Z}_{\mathbb{Z}}$, is (C_1), (C_3) but not (C_2).

Now, we let F be a field and $R = \begin{bmatrix} F & V \\ 0 & F \end{bmatrix}$, where $V = F \oplus F$. We have $eR = \begin{bmatrix} F & V \\ 0 & 0 \end{bmatrix}$, is (C_2) but not (C_1), where $e = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. It is eR indecomposable and not uniform.

To prove (C_1) does not implies (C_2) we take $R = \begin{bmatrix} F & F \\ 0 & F \end{bmatrix}$, where F is a field.

We also known that (C_2) condition implies the (C_3) condition.

Lemma 1 *Let B be a complement of submodule A in M. The following hold:*

(1) $A \oplus B \leq^* M$.

(2) $A \oplus B/B \leq^* M/B$.

Proof. (1) Any nonzero submodule X of M, we need to show that $(A \oplus B) \cap X \neq 0$. It is clear in the case of $X \subseteq B$. Otherwise the maximality of B showing that $(A \oplus B) \cap X \neq 0$. And hence we have $0 \neq a = b + x$ where $a \in A, b \in B, x \in X$.
It is showing that $0 \neq x \in (A \oplus B) \cap X$ since $A \cap B = 0$.

(2) Take any nonzero submodule Y/B of M/B. It means that Y is a submodule of M and Y=B.Suppose that $Y/B \cap (A \oplus B)/B = 0$. Since $Y \neq B, Y \cap A \neq 0$. There is nonzero element $a \in Y \cap A$. Then $a + B \in \frac{Y}{B} \cap \frac{A \oplus B}{B} = 0$. It implies $a \in B$. But $0 = a \in B \cap A = 0$. This is a contradiction. □

We call a submodule N of M is a close submodule if it has no proper essential extensions in M.

Proposition 1. *Let X be an invariant submodule of* $M = M_1 \oplus M_2$, *then* $X = (X \cap M_1) \oplus (X \cap M_2)$ *where* $(X \cap M_i)$ *is an invariant submodule of* M_i.

Lemma 2. *Let M be a right R-module. The following are clear:*

(i) Let M_i *be semisimple fully invariant submodules of M for all* $i \in I$. *Then* $\bigoplus_{i \in I} M_i$ *and* $\bigcap_{i \in I} M_i$ *are semisimple fully invariant submodules of M.*

(ii) If $M = \bigoplus_{i\in I} M_i$ *and X is a semisimple fully invariant submodule of M, then* $X = \bigoplus_{i\in I} \pi_i(X) = \bigoplus_{i\in I} (M_i \cap X)$ *where* π_i *is the ith canonical projection homomorphism of M.*

(iii) Let A, B be submodules of M. If A is a semisimple fully invariant of B, where B is a semisimple fully invariant of M then A must be semisimple fully invariant in M.

Lemma 3. *M is a (strongly) FI−extending module if and only if every (semisimple) fully invariant submodule of M has a complement which is a direct summand of M.*

Proof. Let *X* be a fully invariant submodule and essential in a direct summand of *M*. There exists an idempotent $e \in End(M_R)$ such that *X* is essential in *eM.* And then *(1 − e)M* is the desired complement.

Conversely, let *eM* be a complement of *X,* where idempotent $e \in End(M_R)$. Take any $x \in X$. Then $x = ex + (1-e)x$. Since *X* is a fully invariant submodule of M, $ex \in X \cap eM = 0$. Thus that $X \leq (1-e)M$, and so *X* is essential in *(1 − e)M.* □

Lemma 4. *Let M be a right R-module such that every semisimple fully invariant submodule of M is essential in a direct summand of M. Let X be a fully invariant submodule of M with essential socle. Then X is FI-extending.*

Proof. Let *Y* be a semisimple fully invariant submodule of *X*. By Lemma 2, *Y* is semisimple fully invariant in *M.* Hence, there is a direct summand *N* of *M* such that *Y* is essential in *N.* Let $\pi : M \to Y$ be the projection endomorphism. Then $Y = \pi(Y) \leq \pi(X) \cap D = \pi(X)$. Hence, *Y* is essential in *π(X)* and *π(X)* is a direct summand of *X*. Now, we take any *K* be a fully invariant submodule of *X*. We have *Soc(K)* is a semisimple fully invariant submodule of *K*. By above proof, there exists a direct summand *L* of *K* such that *Soc(K)* is essential in *L*. We suppose that $X = L \oplus L'$ for some submodule *L'* of *X*. It is clear that *Soc(K)* is essential in *K*. Hence, $K \cap L' = 0$. Thus, $Soc(K) \oplus L' \leq K \oplus L'$, and $Soc(K) \oplus L'$ is essential in *K*. It implies that $K \oplus L'$ is essential in *X*. From here, we can deduce what must be proven.

Theorem 2. *Let M be a right R-module, satisfying the possess as follow:*

(I) For submodules X,Y of M, $X \leq^* Y$ *if and only if* $I_X \leq^* I_Y$;

(II) For right ideals K, L of S=End(M), $K \leq^* L$ *if and only if),* $KM \leq^* LM$.

Then the following assertions hold.

(1) If M is CS, then S is right CS.

(2) If M is (strongly) FI-extending, then S is right (strongly) FI-extending.

(3) If M is min CS, then S is right min CS.

(4) If M is a self-generator and max CS (resp. max-min CS), then S is right max CS (resp. right max-min CS).

Proof.

(1) Let *M* be a *CS* module. For any closed right ideal K of S, KM is essential in a direct summand *e(M)* of *M*, where $e = e^2 \in S$. By Lemma 5, we get $K \leq^* I_{KM}$. By *(I)*, we get $I_{KM} \leq^* I_{e(M)} = eS$. Since *K* is closed, we obtain *K* = *eS*, a direct summand of *S*. Consequently, *S* is right *CS*.

(2) Let M be a (strongly) FI-extending module and let K be an ideal of S. Then, KM is a fully invariant submodule of M. Since M is a (strongly) FI-extending module, KM is essential in a (fully invariant) direct summand of M, namely $e(M)$ for some $e = e^2 \in S$. By Lemma 5, we have $K \leq^* I_{KM}$ and $I_{KM} \leq^* I_{e(M)} = eS$ by the condition *(I)*. By closeness of K, we get $K = eS$, a (fully invariant) direct summand of S. This shows that S is right (strongly) FI-extending.

(3) Let M be a min CS module. For a minimal closed (or uniform closed) right ideal K of S, we have $K = I_{KM}$. For nonzero submodules U, V of KM, since M is retractable, I_U and I_V are nonzero. It is clear that I_U and I_V are contained in $I_{KM} = K$, thus $I_U \cap I_V \neq 0$. Then, there exists $0 \neq s \in I_U \cap I_V$, whence, $0 \neq s(M) \subset (U \cap V)$. Therefore, KM is uniform. Since M is min CS, KM is essential in a direct summand $e(M)$ of M, where $e = e^2 \in S$. We have $K \leq^* I_{KM}$, and $I_{KM} \leq^* I_{e(M)} = eS$ by the condition *(I)*. Since K is closed, $K = eS$, a direct summand of S. Consequently, S is right min CS.

(4) Let M be a max CS module and K be maximal closed right ideal of S with $l_S(K) \neq 0$. If KM is essential in a submodule X of M, then $K \leq^* I_{KM} = I_X$. Thus, we have $K = I_{KM} = I_X$ by closeness of K. Since M is a self-generator, we get $K(M) = I_X(M) = X$. This means that KM is a closed submodule and $l_S(KM) \neq 0$. Let Y be a closed submodule containing KM. Then we observe that $K = I_{KM} \subset I_Y$ and I_Y is a closed right ideal by Theorem 1. Thus, $K = I_Y$ by the maximality of K, so $KM = I_Y(M) \leq^* Y$ and $KM = Y$. Consequently, KM is maximal closed. Since M is max CS, KM is a direct summand of M, writing $KM = e(M)$ for some $e = e^2 \in S$. Thus, $K = I_{KM} = I_{e(M)} = eS$, and hence S is right max CS.

The case of max-min CS property is clear. □

Theorem 2. *Let M be a module with the endomorphism ring S. If M has (I) and (II), then S is a Baer (resp. quasi-Baer, right principally quasi-Baer, right Rickart) ring if and only if every M−annihilator of any subset (resp. ideal, principal ideal, element) of S is a direct summand of M.*

Proof. Let S be a Baer ring, and $X = r_M(H)$ be an M−annihilator, for an arbitrary subset $H \subset S$. Then we have $r_S(H) = eS$ for some idempotent $e \in S$, because of Baerness of S. Thus, $HeS = 0$ implies $eM = eS(M) \subset r_M(H)$. For any $x \in r_M(H)$ we have $HxR = 0$ so $HI_{xR} = 0$ and $I_{xR} \subset eS$. Because of (I) and (II), M is retractable and $I_{xR} \neq 0$, hence $xR \cap eM \neq 0$. Consequently, eM is essential in $r_M(H)$ so $eM = r_M(H) = X$.

Now, assume that for every $H \subset S$, $r_M(H)$ is a direct summand of M. Then we have $r_M(H) = eM$ for some $e = e^2 \in S$, and $HeM = HeS(M) = 0$ and $eS \subset r_S(H)$. For any $0 \neq f \in r_S(H)$, $Hf = 0$ so $fM \subset r_M(H) = eM$ and $I_{fM} \subset I_{eM}$. By (I) and (II), we have $eS = I_{eM}$ and $fS \leq^* I_{fM}$, so $fS \cap eS \neq 0$. This implies $eS \leq^* r_S(H)$ so $eS = r_S(H)$. As a consequence, S is a Baer ring. Replacing H by an arbitrary ideal (principal ideal, element) of S, we prove similarly for the other cases of Baer and Rickart conditions.

D.V. Huynh et al. [2, 3] investigated the symmetry of the Goldie and *CS* conditions on a whole prime ring and on one-sided ideals of a prime ring. As a generalization, D.V. Thuat et al. [12] studied the symmetry of Goldie and *CS* conditions on endomorphism rings of finitely generated, quasi-projective self-generators. In this section, we modify the assumptions in [12] to obtain more general results. Indeed, we will remove the assumption of being finitely generated and quasi-projective, even self-generator. The following useful lemma is a combination of [12, Lemma 2.1] and [14, Lemma 3.2].

Lemma 5. *The following statements hold for module M.*

(1) M satisfies the ACC on M−annihilators if and only if S satisfies the ACC on right annihilators.

(2) If M possesses (I) and (II), then u-dim(MR) = n if and only if u-dim(SS) = n, where n is a non-negative integer. Thus, M is Goldie if and only if S is right Goldie.

According to [11, Theorem 2.4 and Theorem 2.9], if *M* is a prime (semiprime) modules, then *S* is a prime (semiprime) ring. Conversely, if *M* is a self-generator and *S* is a prime ring, then *M* is a prime module. In the following lemma, we only need retractability to convert primeness on *S* to *M*.

Lemma 6. *A retractable module M is prime if and only if S is a prime ring.*

Proof. Let *S* be a prime ring and *U*, a fully invariant nonzero submodule of *M*. Then, I_U is a nonzero ideal of *S* because of retractability of *M*. We have $0 \neq I_U(M) \leq U$. For any $f \in S$, if *f(U)* = 0 then $fI_U(M) = fSI_U(M) = 0$ and hence $fSI_U = 0$. Since *S* is a prime ring, the equality *fS* = 0 must be hold. This means f = 0. Thus, the zero is a prime submodule of *M*, so *M* is a prime module. ˆ

Theorem 3. *Let M possess the conditions (I), (II) and (III). Then the following statements are equivalent for some integer n ≥ 2.*

(1) M is prime, Goldie and CS with u-dim(M_R) = n;

(2) S is prime, right Goldie and right CS with u-dim(S_S) = n;

(3) S is prime, left Goldie and left CS with $(u - dim({}_SS) = n$

Proof. The right-left symmetry of the Goldie and *CS* conditions of *S* in (2) ⇔ (3) follows from [2, Theorem 1].

(1)⇔ (2) Firstly, by Lemma 6, *M* is a prime module if and only if *S* is a prime ring. By Lemma 5, *M* is Goldie with *u-dim(S_S) = n* if and only if *S* is right Goldie with *u-dim(S_S) = n*. *M* is a *CS* module if and only if *S* is a right *CS* ring. The proof is completed.

Proposition 2. *Let M possess the conditions (I) and (II). If M is a semiprime, Goldie, CS module, then S is a semiprime, right Goldie, right CS ring. Moreover, we have decomposition for some integer k ≥ 0 :*

- $M = \oplus_{i=1}^{k} M_i$, *where each* M_i *is a prime module, and Hom(*M_i *,* M_j *) = 0 whenever* $i \neq j$,
- $S = \oplus_{i=1}^{k} S_i$, *where each* S_i *= Hom(M, Mi)* S_i ≅ *End(*M_i*) is a prime ring.*

Proof. It is clear that S is semiprime by [11, Theorem 2.9]. Since M is a Goldie, CS module, S is a right Goldie, right CS ring by Theorem 2 and Lemma 7. Thus, S is a direct sum of uniform right ideals, $S = \oplus_{i=1}^{k} e_i S$, where $e_i = e_i S$. We re-arrange the terms to get S = $[e_1 S]\oplus...\oplus[\ e_k S]$, where each $[e_i S]$ is a direct sum of uniform right ideals of the family $\{e_i S\}_{i=1}^{n}$ that are sub-isomorphic to each other, and hence $Hom_S([e_i S],[e_j S]) = 0$ whenever $i \neq j$ for $i, j \in \{1, ..., k\}$. We observe that every $S_i = [e_i S]$ is an ideal of S and is itself a prime ring. We put $M_i = S_i(M)$ for $i = 1, ..., k$. Then, it is clear that $M = \oplus_{i=1}^{k} M_i$ and $Hom(M_i, M_j) = 0$ whenever $i \neq j$ for $i, j \in \{1, ..., k\}$. We see that $S_i \cong End(M_i)$ so M_i is prime, $i \in \{1, ..., k\}$. The proof is now completed.

Example 1. Let F be a field with only two elements and R be a F−algebra having basis $\{e_1, e_2, e_3, n_1, n_2, n_3, n_4\}$ with the following multiplication table

	e_1	e_2	e_3	n_1	n_2	n_3	n_4
e_1	e_1	0	0	0	0	n_3	0
e_2	0	e_2	0	n_1	0	0	n_4
e_3	0	0	e_3	0	n_2	0	0
n_1	n_1	0	0	0	0	0	0
n_2	n_2	0	0	0	0	0	0
n_3	0	0	n_3	0	0	0	0
n_4	0	0	n_4	0	0	0	0

It is easy to see that R is an associative ring with identity $1 = e_1 + e_2 + e_3$. The sum of uniform components $n_1R\oplus n_2R\oplus n_3R\oplus n_4R$ is esential in R_R and R has a decomposition $R = e_1R\oplus e_2R\oplus e_3R$. Thus, R_R is of finite dimension. However, R is not right max CS. In fact, $e_1R\oplus n_1R$ is a maximal closed right ideal (whose left annihilator is $Re_3 \neq 0$) but $e_1R\oplus n_1R$ is not a direct summand. R is also not right min CS, since n_1R is a minimal closed right ideal but not a direct summand.

The module e_1R is not nonsingular because $r_R(n_3) = \text{Span}\{e_1, e_2, e_3, n_1, n_2, n_3, n_4\}$ is essential in R_R. Moreover, e_1R is not a self-generator. In fact, for the simple module $n_3R \leq e_1R$, if $f \in \text{Hom}_R(e_1R, n_3R), f(e_1) = n_3$, then $f(e_1e_1) = f(e_1)e_1 = n_3e_1 = 0$. This implies that $\text{Hom}_R(e_1R, n_3R) = 0$ so e_1R is not retractable, and hence not a self-generator. It is easy to verify that e_1R doesn't satisfy the condition (I). However, e_1R possesses (III) fortunately.

Example 2. [8, Example 3.3] shows that the endomorphism ring of a nonsingular (even projective) CS module may not be right CS. According to [8, Example 3.2], it is possible for a nonsingular (not CS, not retractable) module to have a right CS endomorphism ring. These examples implies that retractability (in particular, the conditions (I) and (II)) in Theorem 2 and Theorem 3 cannot be dropped). This can be generalized for our results on max CS and min CS modules in this paper.

3. Conclusion

Some poperties of fully invariant submodules with CS conditions have founded in this paper. Furthermore, the right-left symmetry of Goldie, CS and max-min CS conditions on the endomorphism rings of prime and semiprime modules have been investigated. Some examples are discussed to guarantee that our main results make sense. There are still many open things that need further research which we want to leave to the readers.

Acknowlegments: *The authors are thankful to referees for their useful comments that improved this paper. It also has the participation of my colleagues and the research group in the ring and module theory led by Professor Nguyen Van Sanh. In which Asso.Prof.Dr Samruam and Dr Thuat Do gave us many good research suggestions. I am thankful to referees for their useful comments that improved this paper.*

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