PULLBACK ATTRACTOR FOR A CLASS OF NON-AUTONOMOUS FUNCTIONAL HYDRODYNAMICAL TYPE EQUATIONS

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Abstract: *This research investigates a specific class of non-autonomous functional hydrodynamical type equations. The focus lies on establishing the existence of a pullback attractor for this system. Pullback attractors are a mathematical concept used to describe the long-term behavior of dynamical systems. In this context, the pullback attractor represents a set that attracts all solutions of the equation as time progresses. By proving the existence of a pullback attractor, the study demonstrates that solutions to these non-autonomous hydrodynamical equations eventually converge to a specific set over time, regardless of their initial conditions. This finding provides valuable insights into the long-term dynamics of the system, which can be helpful in understanding and predicting the behavior of fluid flows governed by such equations.*

Keywords: *Attractor, stochastic hydrodynamical type systems, non-autonomous, uniform attractor.*

1. Introduction

The understanding of the asymptotic behaviour of dynamical systems is one of the most important problems of modern mathematical physics. One way to treat this problem for a system having some dissipativity properties is to analyse the existence and structure of its global attractor, which in the autonomous case, is an invariant compact set which attracts all the trajectories of the system, uniformly on bounded sets. This set has, in general, a very complicated geometry which reflects the complexity of the long-time behaviour of the system (see, for instance, [1, 3, 15, 17], and the references therein).

However, non-autonomous systems are also of great importance and interest as they appear in many applications to natural sciences. On some occasions, some phenomena are modelled by nonlinear evolutionary equations which do not take into account all the relevant information of the real systems. Instead some neglected quantities can be modelled as an external force which in general becomes time-dependent (sometimes periodic, quasiperiodic or almost periodic due to seasonal regimes).

The first attempts to extend the notion of global attractor to the non-autonomous case led to the concept of the so-called uniform attractor (see [4]). It is remarkable that the conditions ensuring the existence of the uniform attractor parallel those for

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autonomous systems. To this end, non-autonomous systems are lifted in [18] to autonomous ones by expanding the phase space. Then, the existence of uniform attractors relies on some compactness property of the solution operator associated to the system. However, one disadvantage of this uniform attractor is that it needs not be "invariant" unlike the global attractor for autonomous systems.

At the same time, the theory of pull-back (or cocycle) attractors has been developed for both the non-autonomous and random dynamical systems (see [4, 5, 7, 9, 11, 13, 16], and has shown to be very useful in the understanding of the dynamics of non-autonomous dynamical systems. In this case, the concept of pullback (or cocycle or non-autonomous) attractor provides a time-dependent family of compact sets which attracts families of sets in a certain universe (e.g. the bounded sets in the phase space) and satisfying an invariance property, which seems to be a natural set of conditions to be satisfied for an appropriate extension of the autonomous concept of attractor. Moreover, this cocycle formulation allows us to handle more general time-dependent terms in the models, not only the periodic, quasi-periodic or almost periodic ones (see, for instance, [4, 5] for non-autonomous models containing hereditary characteristics).

In order to prove the existence of the attractor (in both the autonomous and non-autonomous cases) the simplest, and therefore the strongest, assumption is the compactness of the solution operator associated with the system, which is usually available for parabolic systems in bounded domains. However, this kind of compactness does not hold in general for parabolic equations in unbounded domains and hyperbolic equations on either bounded or unbounded domains. Instead, we often have some kind of asymptotic compactness.

In this situation, there are several approaches to prove the existence of the global (or non-autonomous) attractor. Roughly speaking, the first one ensures the existence of the global (resp. non-autonomous) attractor whenever a compact attracting set (resp. a family of compact attracting sets) exists. The second method consists in decomposing the solution operator (resp. the cocycle or two-parameter semigroup) into two parts: a compact part and another one which decays to zero as time goes to infinity. However, as it is not always easy to find this decomposition, one can use a third approach which is based on the use of the energy equations which are in direct connection with the concept of asymptotic compactness. This third method has been used by Łukaszewicz and Sadowski in [11, 12] (and later also in [13]) to extend to the non-autonomous situation the corresponding one in the autonomous framework (see [10]), but related to uniform asymptotic compactness. Our aim in this paper is to consider the case without uniformity properties and show how the pullback theory works in this situation.

The content of this paper is as follows. In Section 2 we introduce the concept of asymptotically compact non-autonomous dynamical systems. The abstract model will be also stated in this Section. In the last section, we apply theory in Section 2 to prove the existence of a pullback attractor for a abstract model in an unbounded domain in which the external force needs not be bounded. This model not only apply for 2D Navier-Stokes equation but also covers a class of other important equations in fluid mechanic such as 2D magnetohydrodynamic equations, the 2D Bénard magnetic problem, the 3D Leray α-model is quite similar to the Lagrangian averaged Navier-Stokes α-model of turbulence and also shell models of turbulence.

2. Preliminaries

2.1. Pullback attractors

We begin this section by recalling the notion of a pullback attractor. For more details see Lukaszewicz [8] or Carvalho et al. [4]. Let us consider a process U on a metric space X, i.e. a family $\left\{U\left(t,\tau\right):t,\tau,t\geq\tau\right\}$ of continuous mappings $U\left(t,\tau\right):X\rightarrow X$, such that $U\left(t,\tau\right)x=x$, and $U\left(t,\tau\right)=U\left(t,r\right)U\left(r,\tau\right)$ for all $\tau\leq r\leq t$. Let D be a nonempty family of parameterized sets $D=\left\{D\left(t\right):t\in\mathbb{R}\right\}\subset\rho\left(X\right)$, where $\rho\left(X\right)$ denotes the famiy of all nonempty subsets of X.

Definition 2.1. A process $U\left(\cdot,\cdot\right)$ is said to be pullback D-asymptotically compact if for any $t\in\mathbb{R}$ $D\in D$, any sequence $\tau_n\rightarrow-\infty$, any sequence $x_n\in D\left(\tau_n\right)$, the sequence $\left\{U\left(t,\tau_n\right)x_n\right\}$ is relatively compact in X.

Definition 2.2. A family $B\in D$ is pullback D-absorbing for the process $U\left(\cdot,\cdot\right)$ if for any $t\in\mathbb{R}$ and any $D\in D$, there exists $\tau_0\left(t,D\right)\leq t$ such that $U\left(t,\tau\right)D\left(\tau\right)\subset B\left(t\right)$ for $\tau\leq\tau_0\left(t,D\right)$.

Definition 2.3. A family $\Phi=\left\{A\left(t\right):t\in\mathbb{R}\right\}\subset\rho\left(X\right)$ is said to be a pullback D-attractor for the process $U\left(\cdot,\cdot\right)$ if

1. $A\left(t\right)$ is compact for all $t\in\mathbb{R}$,

2. Φ is pullback D-attracting, i.e. $\lim\limits_{\tau\rightarrow-\infty} dist\left(U\left(t,\tau\right)D\left(\tau\right),A\left(t\right)\right)=0$ for all $D\in D$ and all $t\in\mathbb{R}$,

3. Φ is invariant, i.e. $U\left(t,\tau\right)A\left(\tau\right)=A\left(t\right)$ for all $-\infty<\tau\leq t<+\infty$.

The following theorem can be found in [6].

Theorem 2.4. Suppose that a process $U\left(\cdot,\cdot\right)$ is pullback D-asymptotically compact and that $B\in D$ is a family of pullback D-absorbing sets for $U\left(\cdot,\cdot\right)$. Then, the family defined by $\Phi=\left\{A\left(t\right):t\in\mathbb{R}\right\}\subset\rho\left(X\right)$ with $A\left(t\right)=\Omega\left(D,t\right),t\in\mathbb{R}$, where for each $D\in D$

$$\Omega\left(D,t\right)=\bigcap_{s\le t}\left(\overline{\bigcup_{\tau\le s}U\left(t,\tau\right)D\left(\tau\right)}\right)$$

is a pullback D-attractor for $U\left(\cdot,\cdot\right)$ which satisfies in addition that for $t\in\mathbb{R}$, $A\left(t\right)=\overline{\bigcup_{D\in D}\Omega\left(D,t\right)}$. Furthermore, Φ is minimal in the sense that if $\omega=\left\{C\left(t\right):t\in\mathbb{R}\right\}\subset\rho\left(X\right)$ is a family of closed sets such that $\lim_{\tau\to-\infty} dist\left(U\left(t,\tau\right)B\left(\tau\right),C\left(t\right)\right)=0$, then $A\left(t\right)\subset C\left(t\right)$.

2.2. Description of the Model

Let (H, V) denote a separable Hilbert space, A be an (unbounded) self-adjoint positive linear operator on H. Set V = Dom $\left(A^{\frac{1}{2}}\right)$. For $v\in V$ set $\|v\|=\left|A^{\frac{1}{2}}v\right|$. Let V' denote the dual of V (with respect to the inner product $\left(\cdot,\cdot\right)$ of H). Thus we have the Gelfand triple $V\subset H\subset V'$. Let $\left\langle u,v\right\rangle$ denote the duality between $u\in V$ and $v\in V'$ such that $\left\langle u,v\right\rangle=\left(u,v\right)$ for $u\in V, v\in V'$, and let $B:V\times V\to V'$ be a continuous mapping (satisfying the condition **C** given below). The goal of this paper is to study the following abstract model in H:

$\partial_t u\left(t\right)+Au\left(t\right)+B\left(u\left(t\right),u\left(t\right)\right)+Ru\left(t\right)=f\left(t\right)$, where R is a linear bounded operator in H.

We assume that the mapping $B:V\times V\to V'$ satisfies the following antisymmetry and bound conditions:

$\left(C_1\right)$: $B:V\times V\to V'$ is a bilinear continuous mapping.

$\left(C_2\right)$: For $u_i\in V, i=1,2,3,$ we have

$$\left\langle B\left(u_1,u_2\right),u_3\right\rangle=-\left\langle B\left(u_1,u_3\right),u_2\right\rangle \tag{2.1}$$

$\left(C_3\right)$ There exists a Banach (interpolation) space H possessing the properties

i) $V\subset \mathrm{H}\subset H$

ii) there exists a constant a_0 such that

$$a_0\|v\|_{\mathrm{H}}^2\le a_0\left|v\right|\|v\| \tag{2.2}$$

iii) for every $\eta>0$ there exists $C_\eta>0$ such that

$$\left|\left\langle B\left(u_1,u_2\right),u_3\right\rangle\right|\le\eta\|u_3\|^2+C_\eta\|u_1\|_H^2\|u_2\|_{\mathrm{H}}^2 \tag{2.3}$$

Remark 2.5. *The relation in* (2.1) *and* (2.3) *obviously implies that*

$$\left|\left\langle B\left(u_1,u_2\right),u_3\right\rangle\right|\le C\|u_1\|_{\mathrm{H}}\|u_2\|\|u_3\|_{\mathrm{H}} \text{ for } u_i\in V, i=\overline{1,3}. \tag{2.4}$$

3. Pullback attractors for non-autonomous 2D hydrodynamical type systems on some unbounded domain

3.1. Cocycle associated with hydrodynamical type systems

In this section, we construct a θ-cocycle ϕ for the non-autonomous 2D hydrodynamical type systems defined on some unbounded domain *O*. In this section, we derive uniform estimates of solutions of problem (3.1) defined on Ω when $t \to +\infty$. We start with the estimates in *H*. First we assume that $u_0 \in H,\, f \in L^2_{loc}(\mathbb{R}, V')$. For each $\tau \in \mathbb{R}$, we consider the problem:

$$\begin{cases} u \in L^2(\tau, T; V) \cap L^\infty(\tau, T; H) \text{ for all } T > \tau, \\ \dfrac{d}{dt}(u(t), v) + (Au(t), v) + \langle B(u(t), u(t)), v(t) \rangle + (Ru(t); v) = \langle f(t), v(t) \rangle \text{ for all } v \in V, \\ u(\tau) = u_\tau. \end{cases} \tag{3.1}$$

As in the case of bounded domains (see, e.g., [2]), it can be proved that if $f \in L^2_{loc}(\mathbb{R}, V')$ and (2.1)-(2.4) hold true, then problem (3.1) is well-posed in *H*, that is, for every $\tau \in \mathbb{R}$ and $u_\tau \in H$, there exists a unique solution that moreover belongs to $C^0([\tau; +\infty); H)$. To construct a cocycle ϕ for problem (3.1), we denote by Ω = *R*, and define a shift operator θ_t on Ω for every $\tau \in \mathbb{R}$ by $\theta_t(\tau) = t + \tau$. We define the evolutionary process associated to (3.1) as follows. Let us consider the unique solution $u(\cdot, \tau, u_0)$ of problem (3.1). We set

$$U(t, \tau)u_0 = u(t, \tau, u_0), \quad \tau \le t, u_0 \in H. \tag{3.2}$$

From the uniqueness of solution to problem (3.1), one immediately obtains that

$$U(t, \tau)u_0 = U(t, r)U(r, \tau)u_0, \quad \tau \le r \le t, u_0 \in H. \tag{3.3}$$

Also, it is a standard task to prove that for all $\tau \in \mathbb{R}, t \ge 0$ the mapping $U(t, \tau): H \to H$ defined by (3.2) satisfies

$$\left| U(t, \tau)u_0 - U(t, \tau)u_0 \right| \le \left| u_0 - u_0 \right| \exp\left\{ \frac{\lambda}{2\left(\lambda - |R|_{op}\right)} \left(|u_0|^2 + \int_\tau^t \| f(s) \|_*^2 \, ds \right) \right\} \tag{3.4}$$

for any $\tau \le t, u_0, u_0 \in H$: Thus, for all $\tau \le t$, the mapping $U(t; \tau)$ is globally Lipschitz on bounded subsets of *H*, and in particular is a continuous mapping on *H*. Consequently, the family $\{U(t, \tau), t \ge \tau\}$ defined by (3.2) is a process $U(.,.)$ in *H*. Moreover, it can be proved that *U* is weakly continuous, and more exactly the following result holds true. As the proof is identical to that of Lemma 8.1 in [13] but for our particular given by (3.4), we omit it.

Proposition 3.1. *Suppose that* $\{u_0^n\} \subset H$ *is a sequence converging weakly in H to an element* $u_0 \in H$ *. Then*

$$U(t;\ t-\tau)u_{0,n} \to U(t;\ t-\tau)u_0 \text{ weakly in } H \text{ for all } \tau \geq 0;\ t \in \mathbb{R},$$

$$U(\cdot,\tau)u_{0,n} \to U(t;\ t-\tau)u_0 \text{ weakly in } L^2(\tau,T;V) \text{ for all } \tau < T.$$

Let R_σ be the set of all functions $r:\mathbb{R} \to (0;+\infty)$ such that

$$\lim_{t\to-\infty} e^{\sigma t} r^2(t) = 0$$

and denote by D_σ the class of all families $D_\sigma = \{D(t), t \in \mathbb{R}\} \subset \rho(H)$ such that $D(t) \subset B\left(0, r_{D(t)}\right)$ for some D, where $B\left(0, r_{D(t)}\right)$ denotes the closed ball in *H* centered at zero with radius $r_{D(t)}$.

3.2. Existence of pullback attractors

Theorem 3.2. *Under the condition* $f \in L^2_{loc}(\mathbb{R}, V')$ *and* $\int_{-\infty}^{t} \| f(s)\|_*^2\, ds < +\infty$ *for all* $t \in \mathbb{R}$*, then there exists a unique global pullback* D_σ*-attractor belonging to* D_σ *for the process* $U(\cdot,\cdot)$ *given by (3.3).*

Proof. Let $t \in \mathbb{R}, t \geq 0$ and $u \in H$ be fixed, and denote

$$u(r) = u(r;t-\tau;u_0) = \varphi(r-t+\tau;u_0) \text{ for } r \geq t-\tau. \tag{3.5}$$

Taking into account that (*B*(*u; u*)*; u*) = 0, it follows that

$$\frac{d}{dr} e^{\lambda r} |u(r)|^2 + 2e^{\lambda r} \|u(r)\|^2 + 2e^{\lambda r} (Ru(r), u(r)) = \lambda e^{\lambda r} |u(r)|^2 + 2e^{\lambda r} \langle f(r),\ u(r)\rangle \tag{3.6}$$

Because of the properties of scalar products and (3.6), we have that

$$\frac{d}{dr} e^{\sigma r} |u(r)|^2 + 2e^{\sigma r} \|u(r)\|^2 \leq 2e^{\sigma r} \|R\| |u(r)|^2 + \lambda e^{\sigma r} |u(r)|^2 + 2e^{\sigma r} \langle f(r),\ u(r)\rangle$$

$$\leq (2\|R\| + \lambda)^{\sigma r} |u(r)|^2 + e^{\sigma r} \|u(r)\|^2 + 2e^{\lambda r} \| f(r)\|_*^2.$$

Thus,

$$\frac{d}{dr} e^{\lambda r} |u(r)|^2 + \upsilon e^{\lambda r} \|u(r)\|^2 \leq e^{\lambda r} \| f(r)\|_*^2, \text{ where } \upsilon = \lambda - 2\|R\| - \sigma > 0.$$

This implies that

$$e^{\sigma t} |u(t)|^2 \leq e^{\sigma(t-s)} |u(0)|^2 + \int_{t-\tau}^{t} e^{\lambda s} \| f(s)\|_*^2\, ds \tag{3.7}$$

Let $D \in D_\sigma$ be given. From (3.7), we easily obtain that

$$|\phi(\tau, t-\tau, u_0)|^2 \leq e^{\sigma\tau} r^2_{D(t-\tau)} + e^{\sigma t} \int_{-\infty}^{t} e^{\lambda s} \| f(s)\|_*^2\, ds \tag{3.8}$$

for all $u_0 \in D(t-\tau), t \in \mathbb{R}, \tau \geq 0$. Now, denote by $R_\sigma(t)$ the nonnegative number given for each $t \in \mathbb{R}$ by

$$R_\sigma = e^{\sigma t} \int_{-\infty}^{t} e^{\lambda s} \left\| f(s) \right\|_*^2 ds \tag{3.9}$$

and consider the family B_σ of closed balls in H defined by

$$B_\sigma = \left\{ v \in H : |v| \leq R_\sigma(t) \right\} \tag{3.10}$$

It is straightforward to check that $B_\sigma \in D_\sigma$, and moreover, by $\int_{-\infty}^{t} \left\| f(s) \right\|_*^2 ds < +\infty$ and (3.8), the family B_σ is pullback D_σ-absorbing for the cocycle ϕ. Thus, to finish the proof of the theorem we only have to prove that ϕ is pullback D_σ-asymptotically compact. Let us fix $B_\sigma \in D_\sigma$; $t \in \mathbb{R}$, a sequence $\tau_n \to +\infty$, and a sequence $u_{0,n} \in D(t-\tau_n)$. We must prove that from the sequence $\phi(\tau_n, t-\tau_n, u_{0,n})$ we can extract a subsequence that converges in H. As the family B_σ is pullback D_σ-absorbing, for each integer $k \geq 0$ there exists a $\tau_D(k) \geq 0$ such that

$$\phi(\tau, t-\tau-k, D(t-\tau-k)) \subset B_\sigma(t-k) \text{ for all } \tau \geq \tau_D(k). \tag{3.11}$$

Observe now that for $\tau \geq \tau_D(k) + k$ it follows from (3.11) that

$$\phi(\tau-k, t-\tau, D(t-\tau)) \subset B_\sigma(t-k) \tag{3.12}$$

It is not difficult to conclude from (3.12), by a diagonal procedure, the existence of a subsequence $\left\{\tau_{n'}, u_{0,n'}\right\} \subset \left\{\tau_n, u_{0,n}\right\}$, and a sequence $\mathrm{w}_k \in H, k \geq 0$, such that for all $k \geq 0$ and

$$\mathrm{w}_k \in B_\sigma(t-k),\ \phi\left(\tau_{n'} - k, t-\tau_{n'}, u_{0,n'}\right) \to \mathrm{w}_k \text{ weakly in } H \tag{3.13}$$

Observe that, by **Proposition 3.1**, we have

$$\begin{aligned} w_0 &= weak - \lim_{n' \to +\infty} \phi\left(\tau_{n'}, t-\tau_{n'}, u_{0,n'}\right) \\ &= weak - \lim_{n' \to +\infty} \phi\left(k, t-k, \left(weak - \phi\left(\tau_{n'} - k, t-\tau_{n'}, u_{0,n'}\right)\right)\right) \\ &= \phi\left(k, t-k, \left(weak - \phi\left(\tau_{n'}, t-\tau_{n'}, u_{0,n'}\right)\right)\right). \end{aligned} \tag{3.14}$$

This means

$$\phi(k, t-k, \mathrm{w}_k) = \mathrm{w}_0 \text{ for all } k \geq 0. \tag{3.15}$$

Then, by the lower semi-continuity of the norm,

$$\liminf_{n'\to\infty}\left|\phi\left(\tau_{n'},t-\tau_{n'},u_{0,n'}\right)\right|\geq \mathrm{w}_0 \tag{3.16}$$

If we now prove that also

$$\limsup_{n'\to\infty}\left|\phi\left(\tau_{n'},t-\tau_{n'},u_{0,n'}\right)\right|\leq \mathrm{w}_0, \tag{3.17}$$

then we will have

$$\lim_{n'\to\infty}\phi\left(\tau_{n'},t-\tau_{n'},u_{0,n'}\right)=\mathrm{w}_0 \tag{3.18}$$

and this, together with the weak convergence, will imply the strong convergence in *H* of $\phi\left(\tau_{n'},t-\tau_{n'},u_{0,n'}\right)$. In order to prove (3.18), we consider the Hilbert norm in *V* given by

$$[u]^2=\|u\|^2-\left(\|R\|+\frac{\sigma}{2}\right)|u|^2, \tag{3.19}$$

which is equivalent to the norm *u* in *V*.

Then, it is immediate that for all $t\in\mathbb{R},\tau\geq 0$ and all $u_0\in H$,

$$\left|\phi\left(t,t-\tau,u_{0'}\right)\right|^2=e^{-\sigma\tau}|u_0|^2+2\int_{t-\tau}^{t}e^{t-s}\left\langle f(s),\phi(s-t+\tau,t-\tau,u_0)\right\rangle ds$$
$$-2\int_{t-\tau}^{t}e^{s-t}\left[\phi(s-t+\tau,t-\tau,u_0)\right]ds. \tag{3.20}$$

Thus, for all $k\geq 0$ and all $\tau_{n'}\geq k$,

$$\left|\phi\left(\tau_{n'},t-\tau_{n'},u_{0,n'}\right)\right|^2=\left|\phi\left(k,t-k,\phi\left(\tau_{n'},t-\tau_{n'},u_{0,n'}\right)\right)\right|^2=e^{-\sigma\tau}\left|\phi\left(\tau_{n'}-k,t-\tau_{n'},u_{0,n'}\right)\right|^2 \tag{3.21}$$
$$+2\int_{t-\tau}^{t}e^{s-t}\left\langle f(s),\phi\left(s-t+k,t-k,\phi\left(\tau_{n'},t-\tau_{n'},u_{0,n'}\right)\right)\right\rangle ds$$
$$-2\int_{t-\tau}^{t}e^{s-t}\left[\phi\left(s-t+k,t-k,\phi\left(\tau_{n'},t-\tau_{n'},u_{0,n'}\right)\right)\right]^2 ds.$$

By (3.12), we get

$$\phi\left(\tau_{n'}-k,t-\tau_{n'},u_{0,n'}\right)\in B_\sigma(t-k)\ \text{for all}\ \tau_{n'}\geq\tau_D(k)+k,k\geq 0 \tag{3.22}$$

Thus,

$$\limsup_{n'\to\infty}\left(e^{-\sigma k}\left|\phi\left(\tau_{n'}-k,t-\tau_{n'},u_{0,n'}\right)\right|\right)\leq e^{-\sigma k}R_\sigma^2(t-k),k\geq 0. \tag{3.23}$$

On the other hand, as $\phi\left(\tau_{n'}-k,t-\tau_{n'},u_{0,n'}\right)\to \mathrm{w}_k$ weakly in H, by **Proposition 3.1**, we have

$$\phi\left(\cdot-t+k,t-k,\phi\left(\tau_{n'},t-\tau_{n'},u_{0,n'}\right)\right)\to\phi\left(\cdot-t+k,t-k,\mathrm{w}_k\right) \tag{3.24}$$

weakly in $L^2\left(t-k,T;V\right)$. Taking into account that, in particular $e^{\sigma(s-t)} f\left(s\right) \in L\left(t-k,T;V'\right)$, we obtain

$$\lim_{n'\to\infty} \int_{t-\tau}^{t} e^{s-t} \left\langle e^{s-t} f\left(s\right), \phi\left(s-t+k,t-k,\mathrm{w}_k\right)\right\rangle ds = 2\int_{t-\tau}^{t} e^{s-t} \left\langle f\left(s\right), \phi\left(s-t+\tau,t-\tau,u_0\right)\right\rangle ds \tag{3.25}$$

Moreover, as $\left(\int_{t-\tau}^{t} e^{\sigma t}\left[v\left(s\right)\right]^2 ds\right)^{\frac{1}{2}}$ defines a norm in $L^2(t - k, t; V)$, which is equivalent to the usual one, we also obtain from (3.24)

$$\int_{t-\tau}^{t} e^{s-t}\left[\phi\left(s-t+\tau,t-\tau,u_0\right)\right] ds \le \liminf_{n'\to\infty} \int_{t-\tau}^{t} e^{s-t}\left[\phi\left(s-t+k,t-k,\phi\left(\tau_{n'}-k,t-\tau_{n'},u_{0,n'}\right)\right)\right] ds \tag{3.26}$$

By combining (3.21), (3.23), (3.25) and (3.26), we can find that

$$\limsup_{n'\to\infty}\left(e^{-\sigma k}\left|\phi\left(\tau_{n'}-k,t-\tau_{n'},u_{0,n'}\right)\right|\right) \le e^{-\sigma k} R_\sigma^2\left(t-k\right) + 2\int_{t-\tau}^{t} e^{t-s}\left\langle f\left(s\right), \phi\left(s-t+k,t-k,\mathrm{w}_k\right)\right\rangle ds$$
$$-2\int_{t-\tau}^{t} e^{s-t}\left[\phi\left(s-t+k,t-k,\mathrm{w}_k\right)\right] ds. \tag{3.27}$$

Now, from (3.15) and (3.24), we have that

$$\left|\mathrm{w}_0\right|^2 = \left|\phi\left(s-t+k,t-k,\mathrm{w}_{k'}\right)\right|^2 = e^{-\sigma k}\left|\mathrm{w}_k\right|^2 + 2\int_{t-\tau}^{t} e^{t-s}\left\langle f\left(s\right), \phi\left(s-t+k,t-k,\mathrm{w}_{k'}\right)\right\rangle ds \tag{3.28}$$
$$-2\int_{t-\tau}^{t} e^{s-t}\left[\phi\left(s-t+k,t-k,\mathrm{w}_{k'}\right)\right]^2 ds.$$

From (3.27) and (3.28), we have

$$\limsup_{n'\to\infty}\left(\left|\phi\left(\tau_{n'}-k,t-\tau_{n'},u_{0,n'}\right)\right|\right) \le e^{-\sigma k} R_\sigma^2\left(t-k\right) + \left|\mathrm{w}_0\right|^2 - e^{-\sigma k}\left|\mathrm{w}_k\right|^2$$
$$\le e^{-\sigma k} R_\sigma^2\left(t-k\right) + \left|\mathrm{w}_0\right|^2,$$

and thus, taking into account that

$$e^{-\sigma k} R_\sigma^2\left(t-k\right) = 2e^{\sigma t}\int_{-\infty}^{t-k} e^{-\sigma s}\left\|f\left(s\right)\right\|_*^2 ds \to 0$$

as $k \to +\infty$. Thus, we easily obtain (3.21) from the last inequality.

4. Conclusion

This paper has investigated the existence and properties of pullback attractors for a class of non-autonomous functional hydrodynamical type equations. By employing Ball's energy equations, we have established the existence of a unique pullback attractor under suitable conditions on the nonlinear terms and external forces. The results obtained extend the current understanding of the long-time behavior of such complex systems. Further research could explore the structure and dimension of the attractor, as well as its sensitivity to variations in the system parameters.

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