ADMISSIBLE INERTIAL MANIFOLDS FOR ABSTRACT NONAUTONOMOUS THERMOELASTIC PLATE SYSTEMS

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Abstract: *In this paper, we prove the existence of admissible inertial manifolds for the nonautonomous thermoelastic plate systems*

$$\begin{cases} u_{tt} - \mu A\theta + A^2 u &= f(t,u) \\ \theta_t + \eta A\theta + \mu A u_t &= 0 \end{cases}$$

when the partial differential operator A is positive definite and self-adjoint with a discrete spectrum and the nonlinear term f satisfies $\varphi -$Lipschitz condition.

Keywords: *Thermoelastic plate, Lyapunov-Perron method, inertial manifold.*

1. Introduction

One of effective approaches to the study of long - time behavior of infinite dimensional dynamical systems is based on the concept of inertial manifolds which was introduced by C. Foias, G. Sell and R. Temam (see [4] and the references therein). These inertial manifolds are finite dimensional Lipschitz ones, attract trajectories at exponential rate. This enables us to reduce the study of infinite dimensional systems to a class of induced finite dimensional ordinary differential equations.

In this paper, on the real separable Hilbert space $\mathcal{H}$, we study the existence of admissible inertial manifolds of the nonautonomous thermoelastic plate systems:

$$\begin{cases} u_{tt} - \mu A\theta + A^2 u &= f(t,u) \\ \theta_t + \eta A\theta + \mu A u_t &= 0 \end{cases} \tag{1.1}$$

with initial data $u(0) = u_0, u_t(0) = u_1, \theta(0) = \theta_0$.

Here, μ, η are positive constants, A is a positive definite, self-adjoint operator with a discrete spectrum; i.e., there exists the orthonormal basis $\{e_k\} \in \mathcal{H}$ such that $Ae_k = \lambda_k e_k,\ 0 < \lambda_1 \le \lambda_2 \le ...,$ each with finite multiplicity and $\lim_{k\to\infty} \lambda_k = \infty.$

Futhermore, f be a φ - Lipschitz function which is defined as in Definition 2.7.

2. Admissible inertial manifolds

2.1. The fundamental concepts of function spaces and admissibility

Now, we first recall some notions on function spaces and refer to [8] for concrete applications. Denote by $\mathcal{B}$ the Borel algebra and by λ the Lebesgue measure on $\mathbb{R}$. The space $L_{1,loc}(\mathbb{R})$ of real-valued locally integrable functions on $\mathbb{R}$ (modulo λ - nullfunctions)

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becomes a Fréchet space for the seminorms $p_n(f)=\int_{J_n}|f(t)|\,dt$, where $J_n=[n,n+1]$ for each $n\in\mathbb{Z}$ [8].

Definition 2.1. A vector space E of real-valued Borel-measurable functions on $\mathbb{R}$ (modulo λ - nullfunctions) is called a *Banach function space* (over $(\mathbb{R},B,\lambda)$) if

i) E is a Banach lattice with respect to the norm $\|\cdot\|_E$, i.e., $(E,\|\cdot\|_E)$ is a Banach space, and if $\varphi\in E$, ψ is a real-valued Borel-measurable function such that $|\varphi(\cdot)|\le|\psi(\cdot)|$ (λ-a.e.) then $\psi\in E$ and $\|\psi\|_E\le\|\varphi\|_E$,

ii) the characteristic functions χ_A belongs to E for all $A\in\mathcal{B}$ of finite measure and

$$\sup_{t\in\mathbb{R}}\left\|\chi_{[t,t+1]}\right\|_E<\infty,\ \inf_{t\in\mathbb{R}}\left\|\chi_{[t,t+1]}\right\|_E>0,$$

iii) E $\mapsto$ $L_{1,loc}(\mathbb{R})$.

Definition 2.2. Let E be a Banach function space and X be a Banach space endowed with the norm $\|\cdot\|$.

We set $\varepsilon:=\varepsilon(\mathbb{R},X):=\{h:\mathbb{R}\to X\,|\,his\ strongly\ measurable\ and\|h(.)\|\in E\}$ endowed with the norm $\|h\|_\varepsilon:=\|\ \|h(.)\|\ \|_E$.

One can easily see that E is a Banach space. We call it the *Banach space corresponding to the Banach function space* E. We now recall the notion of admissibility [5, 6].

Definition 2.3. The Banach function space E is called *admissible* if it satisfies

i) there is a constant $M\ge1$ such that for every compact interval $[a,b]\subset\mathbb{R}$, we have

$$\int_a^b|\varphi(t)|\,dt\le\frac{M(b-a)}{\left\|\chi_{[a,b]}\right\|_E}\|\varphi\|_E, \tag{1.1}$$

ii) for $\varphi\in E$ the function

$$\Lambda_1\varphi(t)=\int_{t-1}^t\varphi(\tau)d\tau \tag{1.2}$$

belongs to E,

iii) the space E is T_τ^+-invariant and T_τ^--invariant where T_τ^+ and T_τ^- are defined, for $\tau\in\mathbb{R}$, by

$$T_\tau^+\varphi(t):=\varphi(t-\tau)\text{ for t}\in\mathbb{R} \tag{1.3}$$

$$T_\tau^-\varphi(t):=\varphi(t+\tau)\text{ for t}\in\mathbb{R} \tag{1.4}$$

Moreover, there are constants M_1 and M_2 such that

$$\left\|T_\tau^+\right\|\le M_1\quad\text{and}\quad\left\|T_\tau^-\right\|\le M_2\qquad\text{for all }\tau\in\mathbb{R}.$$

We next define the associate spaces of admissible Banach function spaces on $\mathbb{R}$ as follows.

Definition 2.4. Let E be an admissible Banach function space and denote by S(E) the unit sphere in E. Recall that

$$L_1 = \{g : \mathbb{R} \to \mathbb{R} | g \text{ is measurable and } \int_{\mathbb{R}} |g(t)| dt < \infty\}$$

Then, we consider the set E' of all measurable real-valued functions ψ on $\mathbb{R}$ such that

$$\varphi\psi \in L_1, \quad \int_{\mathbb{R}} |\varphi(t)\psi(t)| \, dt \le k \qquad \text{for all } \varphi \in S(E),$$

where k depends only on ψ. Then, E' is a normed space with the norm given by (see [8]):

$$\|\psi\|_{E'} := \sup\left\{\int_{\mathbb{R}} |\varphi(t)\psi(t)| \, dt : \varphi \in S(E)\right\} \quad \text{for all } \psi \in E'.$$

We call E' the *associate space* of E.

Remark 2.5. Let E be an admissible Banach function space and E' be its associate space. Then, from [8. Chapter 2] we also have that the following "Holder's inequality" holds

$$\int_{\mathbb{R}} |\varphi(t)\psi(t)| dt \le \|\varphi\|_E \|\psi\|_{E'} \quad \text{for all } \varphi \in E, \psi \in E'. \tag{1.5}$$

Morever, throughout this paper we need the following assumption

Assumption 1. *The Banach function space E and its associate space E' are admissible spaces. Futhermore, for φ be a positive function belonging to E and any fixed $\nu > 0$ the function $h_\nu(\cdot)$ defined by $h_\nu(t) := \left\|e^{-\nu|t-\cdot|}\varphi(\cdot)\right\|_{E'}$ for $t \in \mathbb{R}$ belongs to E.*

Remark 2.6. In the concept of admissible spaces we can replace whole line $\mathbb{R}$ by an interval $(-\infty, t_0]$.

Definition 2.7. (φ-Lipschitz function). Let E be an admissible Banach function space on $\mathbb{R}$ and φ be a positive function belonging to E. Then, a function $f : \mathbb{R} \times \mathcal{H} \to \mathcal{H}$ is said to be φ *-Lipschitz* if f satisfies

i) $\|f(t,u)\| \le \varphi(t)(1 + \|u\|)$ for a.e. $t \in \mathbb{R}$ and for all $u \in \mathcal{H}$,

ii) $\|f(t,u_1) - f(t,u_2)\| \le \varphi(t)\|u_1 - u_2\|$ for a.e. $t \in \mathbb{R}$ and $\forall u_1, u_2 \in \mathcal{H}$.

2.2. Abstract thermoelastic problem

First, by putting

$$U = \begin{pmatrix} Au \\ u_t \\ \theta \end{pmatrix}, \ \mathcal{A} = A\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -\mu \\ 0 & \mu & \eta \end{pmatrix} = A \cdot G, \ \mathcal{F}(t,U) = \begin{pmatrix} 0 \\ f(t,u) \\ 0 \end{pmatrix}.$$

We can rewrite Equation (1.1) in the form

$$\frac{dU}{dt} + \mathcal{A}U = \mathcal{F}(t,U), \quad t \ge t_0 \tag{1.6}$$

with initial data $U(t_0) = U_0$.

The characteristic polynomial $\chi(z)$ of G has the form $\chi(z)=z^3-\eta z^2+\left(1+\mu^2\right)z-\eta.$

One can see that the equation $\chi(z)=0$ has the simple root z_1 and two other roots are complex $\overline{z}_2=z_3$ such that

$$0<z_1<\eta,\ z_2+z_3=\eta-z_1,\ z_2-z_3=i\cdot\left(\frac{4\eta}{z_1}-\left(\eta-z_1\right)^2\right)^{1/2}$$

if $\dfrac{1}{3}<\rho_1\le\dfrac{1+\mu^2}{\eta^2}\le\rho_2<\infty,$ here ρ_1, ρ_2 are constants.

Moreover, there exists positive constants c_1, c_2 depending on ρ_1, ρ_2 and η_0 such that for any $\eta\ge\eta_0>0$ we have

$$c_1\le z_1\eta\le c_2,\ 1-\frac{c_2}{\eta^2}\le\frac{z_2+z_3}{\eta}<1,\ c_1\le\frac{\left|z_2-z_3\right|}{\eta}\le c_2.$$

In order to diagonalize the matrix operator, we introduce new variables

$$y_1=\frac{\mu\theta-\left(1-z_2z_3\right)Au+\left(z_2+z_3\right)u_t}{\left(z_1-z_2\right)\left(z_1-z_3\right)}$$

$$y_2=\frac{\mu\theta-\left(1-z_1z_3\right)Au+\left(z_1+z_3\right)u_t}{\left(z_2-z_1\right)\left(z_2-z_3\right)}$$

$$y_3=\frac{\mu\theta-\left(1-z_1z_2\right)Au+\left(z_1+z_2\right)u_t}{\left(z_3-z_1\right)\left(z_3-z_2\right)}$$

then

$$\begin{aligned}&Au=y_1+y_2+y_3\\&u_t=-\left(z_1y_1+z_2y_2+z_3y_3\right)\\&\theta=-\frac{1}{\mu}\left(Au+z_1^2y_1+z_2^2y_2+z_3^2y_3\right).\end{aligned}$$

Introducing variables w_j by formulas $y_i(t)=w_i(z_1t)$, we get

$$\begin{cases}\dfrac{dw_1}{dt}+Aw_1 &=K_1f\left(t,A^{-1}\left(w_1+w_2+w_3\right)\right)\\ \dfrac{dw_2}{dt}+\dfrac{z_2}{z_1}Aw_2 &=K_2f\left(t,A^{-1}\left(w_1+w_2+w_3\right)\right)\\ \dfrac{dw_3}{dt}+\dfrac{z_3}{z_1}Aw_3 &=K_2f\left(t,A^{-1}\left(w_1+w_2+w_3\right)\right)\end{cases}\qquad(1.7)$$

where $K_1=\dfrac{z_2+z_3}{z_1\left(z_1-z_2\right)\left(z_1-z_3\right)},\ K_2=\dfrac{z_1+z_3}{z_1\left(z_2-z_1\right)\left(z_2-z_3\right)}$

and

$$K_3 = \frac{z_1 + z_2}{z_1\left(z_3 - z_1\right)\left(z_3 - z_2\right)} = \overline{K}_2.$$

Thus, in the space **H** = H x $\overline{\text{H}}$ x $\overline{\text{H}}$ (where $\overline{H}$ is complexification of $\mathcal{H}$), $W = \left(w_1, w_2, w_3\right)$ satisfies the equation

$$\frac{dW}{dt} + \mathbf{A}W = \mathbf{F}(t,W),\ W(t_0) = W_0 = U_0, \tag{1.8}$$

where

$$\mathbf{A} = A \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{z_2}{z_1} & 0 \\ 0 & 0 & \frac{z_3}{z_1} \end{pmatrix}, \quad \mathbf{F}(t,W) = \begin{pmatrix} K_1 \\ K_2 \\ K_3 \end{pmatrix} \cdot f\left(t, A^{-1}\left(w_1 + w_2 + w_3\right)\right).$$

From now, without any misunderstanding, we denote the norm on **H** by $|\cdot|$ and let $K^2 = K_1^2 + |K_2|^2 + |K_3|^2 = K_1^2 + 2|K_2|^2$ we have

$$\left|\mathbf{F}(t,W)\right| \le \sqrt{3}K\varphi(t)\left(1+|W|\right),\quad \left|\mathbf{F}(t,W_1) - \mathbf{F}(t,W_2)\right| \le \sqrt{3}K\varphi(t)\left|W_1 - W_2\right|. \tag{1.9}$$

In the case of infinite-dimensional phase spaces, instead of (1.8), we consider the integral equation

$$W(t) = e^{-(t-s)\mathbf{A}}W(s) + \int_s^t e^{-(t-\xi)\mathbf{A}}\mathbf{F}(\xi, W(\xi))d\xi \ \text{ for a.e. } t \ge s. \tag{1.10}$$

By a *solution* of equation (1.10) we mean a *strongly measurable* function $W(\cdot)$ defined on an interval J with the values in $\mathcal{H}$ that satisfies (1.10) for $t, s \in J$. We note that the solution W to equation (1.10) is called a *mild solution* of equation (1.8).

2.3. The existence and uniqueness of solution

Now, for every pair of integers $N_1 \ge 0$, and $N_2 \ge 0$ we introduce the projections

$$P = \begin{pmatrix} P_{N_1} & 0 & 0 \\ 0 & P_{N_2} & 0 \\ 0 & 0 & P_{N_2} \end{pmatrix},\ Q = I - P \tag{1.11}$$

where P_N is the orthoprojector onto $\text{span}\{e_k : k = 1,2,...,N\}$ for $N \ge 1$ and $P_0 = 0$.

Putting

$$\lambda^- = \max\left\{\lambda_{N_1}, \frac{\text{Re}z_2}{z_1}\lambda_{N_2}\right\} \quad \text{and} \quad \lambda^+ = \min\left\{\lambda_{N_1+1}, \frac{\text{Re}z_2}{z_1}\lambda_{N_2+1}\right\}$$

Throughout this paper, we assume that $\lambda^- < \lambda^+$. Since $\dim P < \infty$, and P commutes with $\mathbf{A}$, then we have the following dichotomy estimates

$$|e^{tA} P| \le e^{\lambda^- |t|}\ \forall\, t \in \mathbb{R};\ |e^{-tA} Q| \le e^{-\lambda^+ t}\ \forall\, t > 0 \tag{1.12}$$

We now define the *Green function* as follows.

$$\mathcal{G}(t,\tau) = \begin{cases} e^{-(t-\tau)\mathbf{A}}[I-P] & \text{for } t > \tau, \\ -e^{-(t-\tau)\mathbf{A}} P & \text{for } t \le \tau. \end{cases} \tag{1.13}$$

Then $\mathcal{G}(t,\tau)$ maps $\mathbf{H}$ into $\mathbf{H}$. Moreover, by dichotomy estimates (1.12) we have

$$e^{\gamma(t-\tau)} |\mathcal{G}(t,\tau)| \le e^{-\alpha|t-\tau|} \quad \text{for all } t,\tau \in \mathbb{R} \tag{1.14}$$

where $\alpha := \dfrac{\lambda^+ - \lambda^-}{2}$ and $\gamma := \dfrac{\lambda^+ + \lambda^-}{2}$.

Now, by Lyapunov - Perron method, we firstly construct the form of the solutions of equation (1.10) in the following Lemma

Lemma 2.8. *For fixed* $t_0 \in \mathbb{R}$ *let* $W(t)$, $t \le t_0$ *be a solution to equation* (1.10) *such that* $W(t) \in D(\mathbf{A})$ *for all* $t \le t_0$ *and the function* $Z(t) = \left|e^{-\gamma(t_0 - t)} W(t)\right|$ $\forall t \le t_0$, *belongs to* $E_{(-\infty,t_0]}$.

Then, this solution $W(t)$ *satisfies*

$$W(t) = e^{-(t-t_0)\mathbf{A}} v_1 + \int_{-\infty}^{t_0} \mathcal{G}(t,\tau)\mathbf{F}(\tau, W(\tau))d\tau, \ \forall t \le t_0 \tag{1.15}$$

where $v_1 \in P\mathbf{H}$, *and* $\mathcal{G}(t,\tau)$ *is the Green's function defined as in* (1.13).

Proof. Put

$$Y(t) := \int_{-\infty}^{t_0} \mathcal{G}(t,\tau)\mathbf{F}(\tau, W(\tau))d\tau \quad \text{for all } t \le t_0. \tag{1.16}$$

By the definition of $\mathcal{G}(t,\tau)$, we have that $Y(t) \in \mathbf{H}$ for $t \le t_0$.

Using estimates (1.9) and (1.14), for $t \le t_0$, we obtain

$$\begin{aligned} \left|e^{-\gamma(t_0-t)} Y(t)\right| &\le \sqrt{3}K \int_{-\infty}^{t_0} \left|e^{\gamma(t-\tau)} \mathcal{G}(t,\tau)\right| \varphi(\tau) e^{-\gamma(t_0-\tau)} \left(1 + |W(\tau)|\right) d\tau \\ &\le \sqrt{3}K \int_{-\infty}^{t_0} \left|e^{\gamma(t-\tau)} \mathcal{G}(t,\tau)\right| \varphi(\tau) \left(e^{-\gamma(t_0-\tau)} + |W(\tau)|\right) d\tau. \end{aligned} \tag{1.17}$$

Putting $V(t) := e^{-\gamma(t_0-t)} + |W(t)|$ for all $t \le t_0$.

We have that the function V belongs to $E_{(-\infty,t_0]}$ and

$$\begin{aligned} \int_{-\infty}^{t_0} \left|e^{\gamma(t-\tau)} \mathcal{G}(t,\tau)\right| \varphi(\tau) V(\tau) d\tau &\le \int_{-\infty}^{t_0} e^{-\alpha|t-\tau|} \varphi(\tau) V(\tau) d\tau \\ &\le \left\|e^{-\alpha|t-\cdot|} \varphi(\cdot)\right\|_{E'_{(-\infty,t_0]}} \|V\|_{E_{(-\infty,t_0]}}. \end{aligned} \tag{1.18}$$

Here, we use the Holder's inequality (1.5).

Since $h_\alpha(t)=\left\|e^{-\alpha|t-\cdot|}\varphi(\cdot)\right\|_{E'_{(-\infty,t_0]}}$ belongs to $E_{(-\infty,t_0]}$, using the admissibility of $E_{(-\infty,t_0]}$ we obtain that $e^{-\gamma(t_0-\cdot)}Y(\cdot)\in\mathcal{E}_{(-\infty,t_0]}$ and $\left\|e^{-\gamma(t_0-\cdot)}Y(\cdot)\right\|_{\mathcal{E}_{(-\infty,t_0]}}\le\sqrt{3}K\left\|h_\alpha(\cdot)\right\|_{E_{(-\infty,0]}}\left\|V\right\|_{E_{(-\infty,0]}}$.

It is obvious that $Y(\cdot)$ satisfies the integral equation

$$Y(t_0)=e^{-(t_0-t)\mathbf{A}}Y(t)+\int_t^{t_0}e^{-(t_0-\tau)\mathbf{A}}\mathbf{F}(\tau,W(\tau))d\tau\quad\text{for } t\le t_0. \tag{1.19}$$

On the other hand,

$$W(t_0)=e^{-(t_0-t)\mathbf{A}}W(t)+\int_t^{t_0}e^{-(t_0-\tau)\mathbf{A}}\mathbf{F}(\tau,W(\tau))d\tau.$$

Then $Y(t_0)-W(t_0)=e^{-(t_0-t)\mathbf{A}}[Y(t)-W(t)]\in P\mathbf{H}$ and

$$\begin{aligned}W(t)&=e^{-(t-t_0)\mathbf{A}}v_1+Y(t)\\&=e^{-(t-t_0)\mathbf{A}}v_1+\int_{-\infty}^{t_0}\mathcal{G}(t,\tau)\mathbf{F}(\tau,W(\tau))d\tau\quad\text{for } t\le t_0.\end{aligned}$$

The proof is completed.

Lemma 2.9. *Define*

$$h_\alpha(t)=\left\|e^{-\alpha|t-\cdot|}\varphi(\cdot)\right\|_{E'_{(-\infty,t_0]}}. \tag{1.20}$$

Let $f:\mathbb{R}\times\mathbf{H}\to\mathbf{H}$ *be* φ *- Lipschitz such that* $k=\sqrt{3}K\left\|h_\alpha(\cdot)\right\|_{E_{(-\infty,t_0]}}<1.$

Then, there corresponds to each $v_1\in P\mathbf{H}$ *one and only one solution* $W(\cdot)$ *of equation* (1.10) *on* $(-\infty,t_0]$ *satisfying the condition* $PW(t_0)=v_1$ *and* $Z(t)=|e^{-\gamma(t_0-t)}W(t)|,\ t\le t_0$ *belongs to* $E_{(-\infty,t_0]}$ *for each* $t_0\in\mathbb{R}$.

Proof. Denote by $\mathcal{E}^{\gamma,t_0}$ the space of all functions $V:(-\infty,t_0]\to\mathbf{H}$ which is strongly measurable and $\left|e^{-\gamma(t_0-\cdot)}V(\cdot)\right|\in E_{(-\infty,t_0]}$. Then, $\mathcal{E}^{\gamma,t_0}$ is a Banach space endowed with the norm

$$\left\|V\right\|_\gamma:=\left\|\,\left|e^{-\gamma(t_0-\cdot)}V(\cdot)\right|\,\right\|_{E_{(-\infty,t_0]}}. \tag{1.21}$$

For each $t_0\in\mathbb{R}$ and $v_1\in P\mathbf{H}$ we will prove that the linear transformation T defined by

$$(TW)(t)=e^{-(t-t_0)\mathbf{A}}v_1+\int_{-\infty}^{t_0}\mathcal{G}(t,\tau)\mathbf{F}(\tau,W(\tau))d\tau\quad\text{for } t\le t_0 \tag{1.22}$$

acts from $\mathcal{E}^{\gamma,t_0}$ into itself and is a contraction.

In fact, for $W\in\mathcal{E}^{\gamma,t_0}$, we have that $|\mathbf{F}(t,W(t))|\le\sqrt{3}K\varphi(t)\left(1+\left|W(t)\right|\right)$.

Therefore, putting $Y(t):=e^{-(t-t_0)\mathbf{A}}v_1+\int_{-\infty}^{t_0}\mathcal{G}(t,\tau)\mathbf{F}(\tau,W(\tau))d\tau\quad\text{for } t\le t_0$, we derive that

$$\left|e^{-\gamma(t_0-t)}Y(t)\right|\le\left\|v\right\|+\sqrt{3}Kh_\alpha(t)\left\|V\right\|_{E_{(-\infty,t_0]}} \tag{1.23}$$

for all $t\le t_0$, where $V(t):=e^{-\gamma(t_0-t)}(1+|W(t)|)$, and $\left\|v\right\|=e^{-\alpha(t_0-t)}\left\|v_1\right\|$.

Since $e^{-\alpha(t_0-\cdot)}$ and $h_\alpha(\cdot)$ belong to $E_{(-\infty,t_0]}$, $Y(\cdot)\in\mathcal{E}^{\gamma,t_0}$ and $\|Y(.)\|_\gamma \le \|v\| + k\|V\|_{E_{(-\infty,t_0]}}$.

Therefore, the linear transformation T acts from $\mathcal{E}^{\gamma,t_0}$ to $\mathcal{E}^{\gamma,t_0}$.

Now, for $X,Z\in\mathcal{E}^{\gamma,t_0}$ we estimate

$$\left|e^{-\gamma(t_0-t)}\left[TX(t)-TZ(t)\right]\right| \le \int_{-\infty}^{t_0}\left|e^{-\gamma(t_0-t)}\mathcal{G}(t,\tau)\right|\left\|\mathbf{F}(\tau,X(\tau))-\mathbf{F}(\tau,Z(\tau))\right\|d\tau$$

$$\le \sqrt{3}K\int_{-\infty}^{t_0}\left|e^{-\gamma(t_0-t)}\mathcal{G}(t,\tau)\right|\varphi(\tau)e^{-\gamma(t_0-\tau)}\left|X(\tau)-Z(\tau)\right|d\tau.$$

Again, using (1.18) we derive

$$\|TX(.)-TZ(.)\|_\gamma \le \mathrm{k}\|X(.)-Z(.)\|_\gamma.$$

Hence, since $k<1$, we obtain that $T:\mathcal{E}^{\gamma,t_0}\to\mathcal{E}^{\gamma,t_0}$ is a contraction. Thus, there exists a unique $W(\cdot)\in\mathcal{E}^{\gamma,t_0}$ such that $TW=W$. By definition of T we have that $W(\cdot)$ is the unique solution in $\mathcal{E}^{\gamma,t_0}$ of equation (1.10) for $t\le t_0$.

By Lemma 2.9 we proved the existence and uniqueness of solution to Equation (1.10) belongs to $\mathcal{E}^{\gamma,t_0}$ for $t\le t_0$. Futhermore, by Lemma 2.8 this solution can be written in the form of (1.15) which is called *Lyapunov-Perron equation*.

2.4. The existence of admissible inertial manifold

Now, we make precisely the notion of admissible inertial manifolds for solutions to integral equation (1.10) in the following definition.

Definition 2.10. Let E be an admissible function space, $\mathcal{E}$ be a Banach space corresponding to E. An *admissible inertial manifold* of $\mathcal{E}$-class for Equation (1.10) is a collection of Lipschitz surfaces $\mathbb{M}=\{\mathcal{M}_t\}_{t\in\mathbb{R}}$ in $\mathbf{H}$ such that each $\mathcal{M}_t$ is the graph of a Lipschitz function $\mathbf{\Phi}_t : P\mathbf{H}\to(I-P)\mathbf{H},$ i.e.,

$$\mathcal{M}_t=\{U+\mathbf{\Phi}_tU : U\in P\mathbf{H}\} \quad \text{for } t\in\mathbb{R} \tag{1.24}$$

and the following conditions are satisfied:

i) The Lipschitz constants of $\mathbf{\Phi}_t$ are independent of t, i.e. there exists a constant C independent of t such that

$$\left|\mathbf{\Phi}_tW_1-\mathbf{\Phi}_tW_2\right|\le C\,|W_1-W_2| \quad \text{for all } t\in\mathbb{R} \text{ and } W_1,\, W_2\in P\mathbf{H}. \tag{1.25}$$

ii) There exists $\gamma>0$ such that to each $W_0\in\mathcal{M}_{t_0}$ there corresponds one and only one solution $W(t)$ to (1.10) on $(-\infty,t_0]$ satisfying that $W(t_0)=W_0$ and the function

$$V(t)=e^{-\gamma(t_0-t)}W(t) \tag{1.26}$$

belongs to $\mathcal{E}_{(-\infty,t_0]}$ for each $t_0\in\mathbb{R}$.

iii) $\{\mathcal{M}_t\}_{t\in\mathbb{R}}$ is positively invariant under (1.10), i.e., if a solution $W(t),\ t\ge s$ of (1.10) satisfies $W_s\in\mathcal{M}_s$, then we have that $W(t)\in\mathcal{M}_t$ for $t\ge s$.

iv) $\{\mathcal{M}_t\}_{t\in\mathbb{R}}$ exponentially attracts all the solutions to (1.10), i.e., for any solution $W(\cdot)$ of (1.10) and any fixed $s\in\mathbb{R}$, there is a positive constant H such that

$$\operatorname{dist}_{\mathbf{H}}(W(t),\mathcal{M}_t)\le He^{-\gamma(t-s)}\qquad \text{for } t\ge s, \tag{1.27}$$

where γ is the same constant as the one in (1.26), and $\operatorname{dist}_{\mathbf{H}}$ denotes the Hausdorff semi-distance generated by the norm in $\mathbf{H}$.

Then, the existence of admissible inertial manifold is state in the following theorem.

Theorem 2.11. *Equation* (1.10) *has an admissible inertial manifold if*

$$k=\sqrt{3}K\|h_\alpha(\cdot)\|_{E_{(-\infty,t_0]}}<1 \tag{1.28}$$

$$\text{and } \frac{k\sqrt{3}KM_2}{(1-k)(1-e^{-\alpha})}\|\Lambda_1\varphi\|_\infty+k<1, \tag{1.29}$$

where h_α *is given by (1.20) and* M_2 *is defined in* Definition 2.3.

Proof. Firstly, Lemma 2.9 allows us to define a collection of surfaces $\{\mathcal{M}_{t_0}\}_{t_0\in\mathbb{R}}$ by

$$\mathcal{M}_{t_0}:=\left\{V+\mathbf{\Phi}_{t_0}V\ \vdots\ V\in P\mathbf{H}\right\}$$

here $\mathbf{\Phi}_{t_0}:P\mathbf{H}\to(I-P)\mathbf{H}$ is defined by

$$\mathbf{\Phi}_{t_0}(V)=\int_{-\infty}^{t_0}e^{-(t_0-\tau)\mathbf{A}}(I-P)\mathbf{F}(\tau,W(\tau))d\tau=(I-P)W(t_0), \tag{1.30}$$

where $W(\cdot)$ is the unique solution in $\mathcal{E}^{\gamma,t_0}$ of equation (1.10) satisfying that $PW(t_0)=V$ (note that the existence and uniqueness of W is proved in Lemma 2.9).

Then, $\mathbf{\Phi}_{t_0}$ is Lipschitz continuous with Lipschitz constant independent of t_0. Indeed, for V_1 and V_2 belonging to $P\mathbf{H}$ we have

$$\begin{aligned}\left|\mathbf{\Phi}_{t_0}(V_1)-\mathbf{\Phi}_{t_0}(V_2)\right| &\le\int_{-\infty}^{t_0}\left|e^{-(t_0-s)\mathbf{A}}(I-P)\right|\left|\mathbf{F}(s,W_1(s))-\mathbf{F}(s,W_2(s))\right|ds\\ &=\int_{-\infty}^{t_0}\left|\mathcal{G}(t_0,s)\right|\left|\mathbf{F}(s,W_1(s))-\mathbf{F}(s,W_2(s))\right|ds\\ &\le\sqrt{3}K\int_{-\infty}^{t_0}\left|e^{\gamma(t_0-s)}\mathcal{G}(t_0,s)\right|\left|\varphi(s)\right|\left|e^{-\gamma(t_0-s)}\left(W_1(s)-W_2(s)\right)\right|ds\\ &\le k\,|W_1(\cdot)-W_2(\cdot)|_\gamma .\end{aligned} \tag{1.31}$$

We now estimate $|W_1(\cdot)-W_2(\cdot)|_\gamma$. Since $W_i(\cdot)$ is the unique solution in $\mathcal{E}^{\gamma,t_0}$ of equation (1.10) on $(-\infty,t_0]$ satisfying $PW_i(t_0)=V_i$ with $i=1,2$, respectively, we have that

$$\begin{aligned}\left|e^{-\gamma(t_0-t)}\left(W_1(t)-W_2(t)\right)\right|&=\left|e^{-\gamma(t_0-t)}\left(e^{-(t-t_0)\mathbf{A}}(V_1-V_2)+\int_{-\infty}^{t_0}\mathcal{G}(t,\tau)[\mathbf{F}(\tau,W_1(\tau))-\mathbf{F}(\tau,W_2(\tau))]d\tau\right)\right|\\ &\le\left|V_1-V_2\right|+k\,|W_1(\cdot)-W_2(\cdot)|_\gamma\qquad\text{for all } t\le t_0.\end{aligned}$$

Hence, we obtain $|W_1(\cdot)-W_2(\cdot)|_\gamma\le\left|V_1-V_2\right|+k\,|W_1(\cdot)-W_2(\cdot)|_\gamma$.

Therefore, since $k<1$ we get $|W_1(\cdot)-W_2(\cdot)|_\gamma \le \frac{1}{1-k}|V_1-V_2|$.

Substituting this inequality to (1.31) we obtain $\left|\boldsymbol{\Phi}_{t_0}(V_1)-\boldsymbol{\Phi}_{t_0}(V_2)\right| \le \frac{k}{1-k}|V_1-V_2|$,

yielding that $\boldsymbol{\Phi}_{t_0}$ is Lipschitz continuous with the Lipschitz constant $C:=\frac{k}{1-k}$ independent of t_0. We thus obtain the property (i) in Definition 2.10 of the Admissible Inertial Manifold.

Secondly, The property (ii) of the Admissible Inertial Manifold is obvious.

Thirdly, We now prove the property (iii) of admissible inertial manifold.

To do this, let $W(\cdot)$ be a solution to equation (1.10) satisfying $W(s)=W_s\in\mathcal{M}_s$, i.e., $W(s)=PW(s)+\boldsymbol{\Phi}_s(PW(s))$.

Then, we fix an arbitrary number $t_0\in[s,\infty)$ and define a function U on $(-\infty,t_0]$ by

$$U(t)=\begin{cases} W(t) & \text{if } t\in[s,t_0],\\ V(t) & \text{if } t\in(-\infty,s] \end{cases}$$

Where V is the unique solution in $\mathcal{E}^{\gamma,t_0}$ of equation (1.10) satisfying $V(s)=W(s)\in\mathcal{M}_s$.

Then, using equation (1.10) and (1.30) we obtain

$$\begin{aligned} U(t) &= e^{-(t-s)\mathbf{A}}\left(PW(s)+\boldsymbol{\Phi}_s(PW(s))\right)+\int_s^t e^{-(t-\tau)\mathbf{A}}\mathbf{F}(\tau,U(\tau))d\tau \\ &= e^{-(t-s)\mathbf{A}}\left(PW(s)\right)+\int_{-\infty}^t e^{-(t-\tau)\mathbf{A}}(I-P)\mathbf{F}(\tau,U(\tau))d\tau \\ &\quad+\int_s^t e^{-(t-\tau)\mathbf{A}}P\mathbf{F}(\tau,U(\tau))d\tau \quad \text{for } s\le t\le t_0. \end{aligned} \tag{1.32}$$

Obviously, equation (1.32) also remains true for $t\in(-\infty,s]$.

Now, in equation (1.32) setting $t=t_0$ and applying the projection P we obtain

$$PU(t_0)=e^{-(t_0-s)\mathbf{A}}\left(PW(s)\right)+\int_s^{t_0} e^{-(t_0-\tau)\mathbf{A}}P\mathbf{F}(\tau,U(\tau))d\tau \quad \text{for } s\le t_0.$$

It follows from the above equation that

$$\begin{aligned} PW(s) &= e^{(t_0-s)\mathbf{A}}\left(PW(t_0)\right)-\int_s^{t_0} e^{(t_0-s)\mathbf{A}}e^{-(t_0-\tau)\mathbf{A}}P\mathbf{F}(\tau,U(\tau))d\tau \\ &= e^{-(s-t_0)\mathbf{A}}\left(PW(t_0)\right)-\int_s^{t_0} e^{-(s-\tau)\mathbf{A}}P\mathbf{F}(\tau,U(\tau))d\tau \quad \text{for } s\le t_0. \end{aligned} \tag{1.33}$$

Substituting this form of $PU(s)$ to equation (1.32) we obtain

$$U(t)=e^{-(t-t_0)\mathbf{A}}PW(t_0)+\int_{-\infty}^{t_0}\mathcal{G}(t,\tau)\mathbf{F}(\tau,U(\tau))d\tau \quad \text{for } t\le t_0. \tag{1.34}$$

Therefore, $W(t_0)=U(t_0)=PW(t_0)+\boldsymbol{\Phi}_{t_0}\left(PW(t_0)\right)$ for all $t_0\ge s$.

Finally, We prove the property (iv) of Definition 2.10. To do this, we will prove that for any solution $W(\cdot)$ to equation (1.10) and any $s \in \mathbb{R}$ there is a solution $W^{\star}(\cdot)$ of such that $W^{\star}(t) \in \mathcal{M}_t$ for $t \geq s$ and

$$\left|W(t)-W^{\star}(t)\right| \leq \frac{\eta}{1-L} e^{-\gamma(t-s)} \quad \text{for all } t \geq s, \text{ and some constant } \eta, \tag{1.35}$$

where

$$L := \frac{kN_2}{(1-k)(1-e^{-\alpha})}\left\|\Lambda_1\varphi\right\|_{\infty} + k < 1$$

is given as in (1.29). The solution $W^{\star}(\cdot)$ is called an *induced trajectory*.

We will find the induced trajectory in the form $W^{\star}(t) = W(t) + U(t)$ with

$$\|U\|_{s,+} = \operatorname*{esssup}_{t\geq s} e^{\gamma(t-s)}\left|U(t)\right| < \infty. \tag{1.36}$$

Substituting $W^{\star}(\cdot)$ into (2.10) we obtain that $W^{\star}(\cdot)$ is a solution to (1.10) for $t \geq s$ if and only if $U(\cdot)$ is a solution to the equation

$$U(t) = e^{-(t-s)\mathbf{A}}U(s) + \int_s^t e^{-(t-\xi)\mathbf{A}}[\mathbf{F}(\xi, W(\xi)+U(\xi)) - \mathbf{F}(\xi, W(\xi))]d\xi. \tag{1.37}$$

For the sake of simplicity in the presentation, we put

$$F(t,U) = \mathbf{F}(t, W+U) - \mathbf{F}(t,W)$$

and set

$L_{\infty}^{s,+} = \{V : [s,\infty) \to \mathbf{H} \,|\, V$ is strongly measurable and $\operatorname*{esssup}_{t\geq s} e^{\gamma(t-s)}\left|\mathrm{V}(t)\right| < \infty\}$ endowed with the norm $\|.\|_{s,+}$ defined as in (1.36).

Then, by the same way as in Lemma 2.8, we can prove that a function $U(\cdot) \in L_{\infty}^{s,+}$ is a solution to (1.37) if and only if it satisfies

$$U(t) = e^{-(t-s)\mathbf{A}}X_0 + \int_s^{\infty} \mathcal{G}(t,\tau)F(\tau, U(\tau))d\tau \quad \text{for } t \geq s \text{ and some } X_0 \in (I-P)\mathbf{H}. \tag{1.38}$$

Here, the value $X_0 \in (I-P)\mathbf{H}$ is chosen such that $W^{\star}(s) = W(s) + U(s) \in \mathcal{M}_s$, i.e.,

$$(I-P)(W(s)-U(s)) = \mathbf{\Phi}_s\left(P(W(s)+U(s))\right).$$

From (1.38) it follows that

$$U(s) = X_0 - \int_s^{\infty} e^{-(s-\tau)\mathbf{A}} PF(\tau, U(\tau))d\tau. \tag{1.39}$$

Hence

$$P(W(s)+U(s)) = PW(s) - \int_s^{\infty} e^{-(s-\tau)\mathbf{A}} PF(\tau, U(\tau))d\tau,$$

and therefore

$$X_0 = (I-P)U(s) = -(I-P)W(s) + \mathbf{\Phi}_s\left(PW(s) - \int_s^{\infty} e^{-(s-\tau)\mathbf{A}} PF(\tau, U(\tau))d\tau\right). \tag{1.40}$$

Substituting this form of X_0 into (1.38) we obtain

$$U(t) \quad = e^{-(t-s)\mathbf{A}}\left[-(I-P)W(s)+\mathbf{\Phi}_s\left(PW(s)-\int_s^\infty e^{-(s-\tau)\mathbf{A}}PF(\tau,U(\tau))d\tau\right)\right] \qquad (1.41)$$

$$+\int_s^\infty \mathcal{G}(t,\tau)F(\tau,U(\tau))d\tau \quad \text{for } t\geq s.$$

What we have to do now to prove the existence of $W^\star$ satisfying (1.35) is to prove that equation (1.41) has a solution $U(\cdot)\in L_\infty^{s,+}$.

To do this we will prove that the linear transformation T defined by

$$(TX)(t)=e^{-(t-s)\mathbf{A}}\left[-(I-P)W(s)+\mathbf{\Phi}_s\left(PW(s)-\int_s^\infty e^{-(s-\tau)\mathbf{A}}PlF(\tau,X(\tau))d\tau\right)\right]$$

$$+\int_s^\infty \mathcal{G}(t,\tau)\mathcal{F}(\tau,X(\tau))d\tau \quad \text{for } t\geq s,$$

acts from $L_\infty^{s,+}$ into itself and is a contraction.

Indeed, for $X(\cdot)\in L_\infty^{s,+}$, we have that $|F(t,X(t))|\leq\sqrt{3}K\varphi(t)|X(t)|$, therefore, by putting

$$q(X):=-(I-P)W(s)+\mathbf{\Phi}_s\left(PW(s)-\int_s^\infty e^{-(s-\tau)\mathbf{A}}PF(\tau,X(\tau))d\tau\right),$$

we can estimate

$$e^{\gamma(t-s)}\left|(TX)(t)\right| \leq e^{\gamma(t-s)}\left|e^{-(t-s)\mathbf{A}}q(X)\right|+\sqrt{3}K\int_s^\infty\left|e^{\gamma(t-\tau)}\mathcal{G}(t,\tau)\right|\varphi(\tau)e^{\gamma(\tau-s)}\left|X(\tau)\right|d\tau \qquad (1.42)$$

$$\leq\left\|e^{\gamma(t-s)}e^{-(t-s)\mathbf{A}}q(X)\right\|+\sqrt{3}K\int_s^\infty\left|e^{\gamma(t-\tau)}\mathcal{G}(t,\tau)\right|\varphi(\tau)d\tau\left\|X(.)\right\|_{s,+}.$$

Using Lipschitz property of $\mathbf{\Phi}_s$ and for $t\geq s$ we now estimate the first term in the right-hand side of the last formula as follows.

$$\begin{aligned}
\left|e^{\gamma(t-s)}e^{-(t-s)\mathbf{A}}q(X)\right| &\leq \left|e^{\gamma(t-s)}e^{-(t-s)\mathbf{A}}(-(I-P)W(s)+\mathbf{\Phi}_s(PW(s)))\right|+\\
&\quad +\left|e^{\gamma(t-s)}e^{-(t-s)\mathbf{A}}(q(X)+(I-P)W(s)-\mathbf{\Phi}_s(PW(s)))\right|\\
&\leq e^{(\gamma-\lambda^+)(t-s)}\Big(\left|(-(I-P)W(s)+\mathbf{\Phi}_s(PW(s)))\right|+\\
&\quad +\left|(q(X)+(I-P)W(s)-\mathbf{\Phi}_s(PW(s)))\right|\Big)\\
&\leq \eta+\left|(q(X)+(I-P)W(s)-\mathbf{\Phi}_s(PW(s)))\right|\\
&\leq \eta+\left|\mathbf{\Phi}_s\left(PW(s)-\int_s^\infty e^{-(s-\tau)\mathbf{A}}PF(\tau,X(\tau))d\tau\right)-\mathbf{\Phi}_s(PW(s))\right|\\
&\leq \eta+\frac{k}{1-k}\left|\int_s^\infty e^{-(s-\tau)\mathbf{A}}PF(\tau,X(\tau))d\tau\right|\\
&\leq \eta+\frac{k\sqrt{3}K}{1-k}\int_s^\infty e^{-\alpha(\tau-s)}\varphi(\tau)\left|e^{\gamma(\tau-s)}X(\tau)\right|d\tau\\
&\leq \eta+\left[\frac{k\sqrt{3}KM_2}{(1-k)(1-e^{-\alpha})}\left\|\Lambda_1\varphi\right\|_\infty\right]\left\|X(.)\right\|_{s,+}.
\end{aligned}$$

Substituting these estimates to (1.42) we obtain $TX \in L_\infty^{s,+}$ and

$$\|TX\|_{s,+} \le \eta + \left[\frac{k\sqrt{3}KM_2}{(1-k)(1-e^{-\alpha})} \|\Lambda_1\varphi\|_\infty + k \right] \|X(.)\|_{s,+}. \tag{1.43}$$

Therefore, the linear transformation T acts from $L_\infty^{s,+}$ to $L_\infty^{s,+}$.

Now, using the fact that $|F(t,U_1) - F(t,U_2)| \le \sqrt{3}K\varphi(t)|U_1 - U_2|$ and for $X, Z \in L_\infty^{s,+}$ we now estimate

$$\begin{aligned}
\left|e^{\gamma(t-s)}\left(TX(t) - TZ(t)\right)\right| &\le \frac{k}{1-k}\left|\int_s^\infty e^{-(s-\tau)\mathrm{A}} P\big(F(\tau, X(\tau)) - F(\tau, Z(\tau))\big)d\tau\right| \\
&\quad + \int_s^\infty \left|e^{\gamma(t-s)}\mathcal{G}(t,\tau)\right| |F(\tau, X(\tau)) - F(\tau, Z(\tau))| \, d\tau \\
&\le \frac{k\sqrt{3}K}{1-k}\int_s^\infty e^{-\alpha(\tau-s)}\varphi(\tau)\left|e^{\gamma(\tau-s)}\left[X(\tau) - Z(\tau)\right]\right| d\tau \\
&\quad + \sqrt{3}K\int_s^\infty \left|e^{\gamma(t-\tau)}\mathcal{G}(t,\tau)\right| \varphi(\tau) e^{\gamma(\tau-s)} |X(\tau) - Z(\tau)| \, d\tau \\
&\le \left[\frac{k\sqrt{3}KM_2}{(1-k)(1-e^{-\alpha})} \|\Lambda_1\varphi\|_\infty + k \right] \|X(.) - Z(.)\|_{s,+} \quad \text{for all } t \ge s.
\end{aligned}$$

Therefore, $\|TX(.) - TZ(.)\|_{s,+} \le \left[\frac{k\sqrt{3}KM_2}{(1-k)(1-e^{-\alpha})} \|\Lambda_1\varphi\|_\infty + k \right] \|X(.) - Z(.)\|_{s,+}$.

Hence, if $\frac{k\sqrt{3}KM_2}{(1-k)(1-e^{-\alpha})} \|\Lambda_1\varphi\|_\infty + k < 1$ then we obtain that $T : L_\infty^{s,+} \to L_\infty^{s,+}$ is a contraction. Thus, there exists a unique $U(\cdot) \in L_\infty^{s,+}$ such that $TU = U$.

By the definition of T we have that $U(\cdot)$ is the unique solution in $L_\infty^{s,+}$ of equation (1.41) for $t \ge s$. Also, using (1.43) we have

$$\|U(.)\|_{s,+} \le \frac{\eta}{1-L}.$$

Furthermore, by determination of U we obtain the existence of the solution $W^\star = W + U$ to equation (1.10) such that $W^\star(t) \in \mathcal{M}_t$ for $t \ge s$, and $W^\star$ satisfies the inequality (1.35) yielding that $|W^\star(t) - W(t)| = |U(t)| \le \frac{\eta}{1-L} e^{-\gamma(t-s)}$ for all $t \ge s$.

Putting $H := \frac{\eta}{1-L}$ it follows from the latter inequality that

$$\mathrm{dist}_{\mathbf{H}}(W(t), \mathcal{M}_t) \le He^{-\gamma(t-s)} \quad \text{for all } t \ge s.$$

Therefore, $\{\mathcal{M}_t\}_{t\in\mathbb{R}}$ exponentially attracts every solution $W(\cdot)$ of integral equation (1.10).

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