GLOBAL ATTRACTOR FOR A NONLOCAL p-LAPLACE PARABOLIC EQUATION WITH NONLINEARITY OF ARBITRARY ORDER

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Abstract: *In this paper we consider a nonlocal p-Laplace parabolic equation depending on the L^p norm of the gradient with nonlinearity of arbitrary order. First, we prove the existence and uniqueness of weak solutions by combining the compactness and monotone methods and the weak convergence techniques in Orlicz spaces. Then, we prove the existence and regularity of a global attractor for the associated semigroup. The main novelty of our results is that no restriction on the upper growth of the nonlinearity is imposed.*

Keywords: *Nonlocal p-Laplace parabolic equation; nonlinearity of arbitrary order, weak solution, global attractor, compactness method, monotone method, weak convergence techniques.*

1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^N$ with Lipschitz boundary $\partial\Omega$ and let $p \geq 2$ be fixed. We consider the following quasilinear parabolic equation with nonlocal diffusion term

$$\begin{cases} u_t - \operatorname{div}\left(a\left(\|\nabla u\|^p_{L^p(\Omega)}\right)|\nabla u|^{p-2}\nabla u\right) + f(u) = g(x), & x \in \Omega, t > 0, \\ u(x,t) = 0, & x \in \partial\Omega, t > 0, \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{1.1}$$

where the diffusion coefficient a, the nonlinearity f, and the external force g satisfy the following conditions:

(H1) $a \in C(\mathbb{R}, \mathbb{R}_+)$ and there are two positive constants m and M such that

$$0 < m \leq a(s) \leq M, \quad \forall s \in \mathbb{R}. \tag{1.2}$$

Moreover, we assume that

$$s \mapsto a(s^p)s^{p-1} \text{ is nondecreasing.} \tag{1.3}$$

(H2) $f : \mathbb{R} \to \mathbb{R}$ is a continuously differentiable function satisfying

$$f(u)u \geq -\mu u^2 - c_1, \tag{1.4}$$

$$f'(u) \geq -\ell, \tag{1.5}$$

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where μ, c_1, ℓ are positive constants, and if $p=2$ then we assume furthermore that $0<\mu<m\lambda_1$ with $\lambda_1>0$ being the first eigenvalue of the Laplace operator $-\Delta$ in Ω associated with the homogeneous Dirichlet boundary condition.

(H3) $g \in L^2(\Omega)$.

Equation (1.1) is nonlocal due to the structure of the diffusion coefficient which depends upon the L^p -norm of the gradient. In the last decade, a lot of attention has been devoted to nonlocal parabolic problems. One of the justifications of such models lies in the fact that in reality the measurements are not made pointwise but through some local average. Some interesting features of nonlocal parabolic equations and systems and more motivation are described in [1] [6] [7] [8] [9] [22] [27] and references therein.

On the other hand, the existence and long-time behavior of solutions in terms of the existence of global attractors to quasilinear parabolic equations involving p -Laplacian type operators have been extensively studied in recent years. A typical example of nonlinearity is the one satisfying a growth and dissipative condition of polynomial type

$$c_1 |u|^p - c_0 \le f(u)u \le c_2 |u|^p + c_0,$$
$$f'(u) \ge -\ell,$$

for some $p \ge 2$, see e.g. [2, 3, 5, 10, 11, 14, 17, 24, 26]. We notice that this class of nonlinearities requires some restrictions on the upper growth, and in particular, the exponential nonlinearity, for example, $f(u)=e^u$, do not hold.

For nonlocal p -Laplace parabolic equations, in some recent works [8] [9], Caraballo et. al. considered the following equation

$$u_t - a(l(u))\operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) + f(u) = g(x),$$

where $a: L^2(\Omega) \to \mathbb{R}$ is a continuous linear functional, $f(u)$ is sublinear or is growth and dissipative of polynomial type. They proved the existence of global attractors in both cases with and without uniqueness of solutions, and these results are in some sense as extensions of previous ones for the following nonlocal reaction-diffusion equation in [4] [6] [20] [22].

$$u_t - a(l(u))\Delta u + f(u) = g(x).$$

While in [12, 13], Chipot and Savitska considered the following parabolic equation

$$u_t - \operatorname{div}\left(a\left(\| \nabla u \|^p_{L^p(\Omega)}\right)|\nabla u|^{p-2}\nabla u\right) = g(x),$$

with zero Dirichlet boundary conditions, where $g \in W^{-1,q}(\Omega)$. They proved the existence, uniqueness and long-time behavior of solutions to this problem. This result was extended in [25] with the nonlinearity satisfying the dissipative condition of polynomial type.

In this paper, we extend the results in [4, 12, 13, 25] by adding a nonlinearity of arbitrary order. Here we are able to prove the existence and uniqueness of weak solutions and the existence of a global attractor for a very large class of nonlinearities that particular covers both sublinear and polynomial type classes and even exponential nonlinearities. The absence of the upper growth condition on f and the diffusion coefficient determined by a global quantity causes a number of difficulties which make the analysis of the problem interesting. To overcome these essential difficulties, we exploit the weak convergence techniques in Orlicz spaces [17], and combine it with the standard monotone and compactness methods. In particular, the results obtained here are extensions of many previous results in [16] [24] [26] for local p-Laplace parabolic equations.

The paper is organized as follows. In Section 2, we prove the existence and uniqueness of global weak solutions to problem (1.1). In Section 3, we show the existence of global attractors in various spaces for the continuous semigroup associated to problem (1.1).

2. Existence and uniqueness of weak solutions

Let us denote $\Omega_T := \Omega\times(0,T)$ and let (p,q) be conjugate, i.e., $\frac{1}{p}+\frac{1}{q}=1$. First, we give the definition of weak solutions to problem (1.1).

Definition 2.1. *Let $u_0\in L^2(\Omega)$ be given. A function u is called a weak solution of problem (1.1) on the interval $(0,T)$ if $u\in L^p(0,T;W_0^{1,p}(\Omega))\cap C([0,T];L^2(\Omega))$, $\frac{du}{dt}\in L^q(0,T;W^{-1,q}(\Omega))+L^1(\Omega_T)$, $f(u)\in L^1(\Omega_T), u(0)=u_0$, and*

$$\int_\Omega\Big(u_t v + a(\|\nabla u\|^p_{L^p(\Omega)})|\nabla u|^{p-2}\nabla u\cdot\nabla v + f(u)v - gv\Big)dx = 0,$$

for all test functions $v\in W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$ and for a.e. $t\in(0,T)$.

As in [12], under the assumption **(H1)**, one can check that the operator

$$Au := -\mathrm{div}\Big(\mathrm{a}(\|\nabla \mathrm{u}\|^{\mathrm{p}}_{\mathrm{L^p}(\Omega)})|\nabla \mathrm{u}|^{\mathrm{p-2}}\nabla \mathrm{u}\Big)$$

is monotone in $W_0^{1,p}(\Omega)$, i.e., for all $u,v\in W_0^{1,p}(\Omega)$, we have

$$\langle Au-Av, u-v\rangle \ge 0. \tag{2.1}$$

In addition, for each $u\in W_0^{1,p}(\Omega)$, we have the following inequality

$$\lambda_1\|u\|^p_{L^p(\Omega)} \le \|u\|^p_{W_0^{1,p}(\Omega)}. \tag{2.2}$$

In the case $p>2$, it follows from the embedding $L^p(\Omega)\subset L^2(\Omega)$ and the inequality (2.2) that

$$\|u\|^2_{L^2(\Omega)} \le |\Omega|^{\frac{p-2}{p}}\|u\|^2_{L^p(\Omega)} \le |\Omega|^{\frac{p-2}{p}}\lambda_1^{-\frac{2}{p}}\|u\|^2_{W_0^{1,p}(\Omega)}.$$

As an application of the Young inequality with ε, we obtain

$$\|u\|^2_{L^2(\Omega)} \le \varepsilon\|u\|^p_{W_0^{1,p}(\Omega)} + \frac{(p-2)|\Omega|}{p}\left(\frac{p\lambda_1\varepsilon}{2}\right)^{-\frac{2}{p-2}}. \tag{2.3}$$

Using the Holder inequality, inequality (2.2), the Young inequality and the embedding $L^p(\Omega) \subset L^2(\Omega)$, we have the following inequality

$$\left|\int_\Omega gudx\right| \le \varepsilon \| u \|^p_{W_0^{1,p}(\Omega)} + \frac{|\Omega|^{\frac{(p-2)q}{2p}}}{q(p\varepsilon\lambda_1)^{\frac{q}{p}}} \| g \|^q_{L^2(\Omega)}, \tag{2.4}$$

for all $u \in W_0^{1,p}(\Omega)$ and any $\varepsilon > 0$.

Theorem 2.1. *Under the assumptions* ***(H1)-(H3)****, problem (1.1) has a unique global weak solution* u *satisfying*

$$u \in C([0,\infty); L^2(\Omega)) \cap L^p_{loc}(0,\infty; W_0^{1,p}(\Omega)),$$
$$\frac{du}{dt} \in L^q_{loc}(0,\infty; W^{-1,q}(\Omega)) + L^1_{loc}(0,\infty; L^1(\Omega)).$$

Moreover, the mapping $u_0 \mapsto u(t)$ *is continuous on* $L^2(\Omega)$*, that is, the solution depends continuously on the initial data.*

Proof. **i) Existence.** Fix $T > 0$ arbitrarily. Let $\{e_j\}_{j=1}^\infty$ be a basis of $W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$, which is orthornomal in $L^2(\Omega)$. We look for an approximate solution $u_n(t)$ of the form

$$u_n(t) = \sum_{j=1}^n \gamma_{nj}(t) e_j$$

that solves the following problem

$$\begin{cases} \int\limits_\Omega \left[u_{nt} e_k + a\left(\| \nabla u_n \|^p_{L^p(\Omega)} \right) |\nabla u_n|^{p-2} \nabla u_n \cdot \nabla e_k + f(u_n) e_k \right] dx = \int\limits_\Omega g e_k dx, \\ \sum\limits_{k=1}^n \gamma_{nk}(0) e_k \to u_0 \text{ in } \mathrm{L}^2(\Omega) \text{ as } \mathrm{n} \to \infty. \end{cases} \tag{2.5}$$

Since $a \in C(\mathbb{R}, \mathbb{R}_+)$ and $f \in C^1(\mathbb{R})$, the Peano theorem ensures the existence of approximate solutions $u_n(t)$ on an interval $[0, T_n) \subset [0, T]$.

We now establish some *a priori* estimates for u_n. Multiplying the first equation in (2.5) by $\gamma_{nj}(t)$ and summing from $j = 1$ to n, we obtain

$$\frac{1}{2}\frac{d}{dt} \| u_n \|^2_{L^2(\Omega)} + a\left(\| \nabla u_n \|^p_{L^p(\Omega)} \right) \int_\Omega |\nabla u_n|^p \, dx + \int_\Omega f(u_n) u_n dx = \int_\Omega g u_n dx. \tag{2.6}$$

Hence, by (1.2) and (1.4), we have

$$\frac{1}{2}\frac{d}{dt} \| u_n \|^2_{L^2(\Omega)} + m \| u_n \|^p_{W_0^{1,p}(\Omega)} - \mu \| u_n \|^2_{L^2(\Omega)} \le c_1 |\Omega| + \int_\Omega g u_n dx.$$

Putting the above inequality together with (2.2), (2.3) and (2.4), there exist two positive constants C_1, C_2 such that

$$\frac{d}{dt}\| u_n \|^2_{L^2(\Omega)} + C_1 \| u_n \|^p_{W_0^{1,p}(\Omega)} \leq C_2.$$

Integrating from 0 to t, $0 < t \leq T_n$, we get

$$\| u_n(t) \|^2_{L^2(\Omega)} + C_1 \int_0^t \| u_n(s) \|^p_{W_0^{1,p}(\Omega)} ds \leq \| u_0 \|^2_{L^2(\Omega)} + C_2 T.$$

This implies that $\{u_n\}$ is bounded in $L^\infty(0,T_n;L^2(\Omega))$ and in $L^p(0,T_n;W_0^{1,p}(\Omega))$. In particular, we see that $\| u_n(t) \|_{L^2(\Omega)}$ remains bounded in time. Therefore, we can extend the approximate solution to the whole interval $[0,T]$.

On the other hand, for a.e. $t \in (0,T)$,

$$Au_n(t) = -\mathrm{div}\left(a\left(\| \nabla u_n(t) \|^p_{L^p(\Omega)} \right) | \nabla u_n(t) |^{p-2} \nabla u_n(t) \right)$$

defines an element of $W^{-1,q}(\Omega)$ by

$$\langle Au_n(t), v \rangle = a\left(\| \nabla u_n(t) \|^p_{L^p(\Omega)} \right) \int_\Omega |\nabla u_n(t)|^{p-2} \nabla u_n(t) \cdot \nabla v dx,$$

for all $w \in W_0^{1,p}(\Omega)$. Using (1.2) and the boundedness of $\{u_n\}$ in $L^p(0,T;W_0^{1,p}(\Omega))$, we deduce that $\{Au_n\}$ is bounded in $L^q(0,T;W^{-1,q}(\Omega))$ since

$$\begin{aligned}
& \left| \int_0^T \langle -\mathrm{div}(a(\| \nabla u_n \|^p_{L^p(\Omega)}) | \nabla u_n |^{p-2} \nabla u_n), v \rangle dt \right| \\
= \ & \left| \int_{\Omega_T} a(\| \nabla u_n \|^p_{L^p(\Omega)}) | \nabla u_n |^{p-2} \nabla u_n \cdot \nabla v dx dt \right| \\
\leq \ & M \int_{\Omega_T} |\nabla u_n|^{p-1} | \nabla v | dx dt \\
\leq \ & M \| u_n \|^{p/q}_{L^p(0,T;W_0^{1,p}(\Omega))} \| v \|_{L^p(0,T;W_0^{1,p}(\Omega))}
\end{aligned}$$

for any $v \in L^p(0,T;W_0^{1,p}(\Omega))$. We now prove that $\{f(u_n)\}$ is bounded in $L^1(\Omega_T)$. It follows from (1.2), (2.6) and (2.4) that

$$\frac{1}{2}\frac{d}{dt}\| u_n \|^2_{L^2(\Omega)} + \int_\Omega f(u_n) u_n dx \leq C\left(m, p, q, \lambda_1, |\Omega| \right) \| g \|^q_{L^2(\Omega)}.$$

Integrating from 0 to T, we obtain

$$\frac{1}{2}\| u_n(T) \|^2_{L^2(\Omega)} + \int_{\Omega_T} f(u_n) u_n dx dt \leq \frac{1}{2}\| u_n(0) \|^2_{L^2(\Omega)} + TC(m,p,q,\lambda_1,|\Omega|) \| g \|^q_{L^2(\Omega)}.$$

Hence

$$\int_{\Omega_T} f(u_n) u_n dx dt \leq \frac{1}{2}\| u_0 \|^2_{L^2(\Omega)} + TC(m,p,q,\lambda_1,|\Omega|) \| g \|^q_{L^2(\Omega)}. \tag{2.7}$$

Setting $h(u_n) = f(u_n) - f(0) + \nu u_n$ with $\nu > \ell$. By (1.5), it implies that $h(s)s \geq 0$ for all $s \in \mathbb{R}$. Therefore, we deduce from (2.7) and the boundedness of $\{u_n\}$ in $L^\infty(0,T;L^2(\Omega))$ that

$$\begin{aligned}\int_{\Omega_T} |h(u_n)|\,dxdt &\le \int_{\Omega_T\cap\{|u_n|>1\}} |h(u_n)u_n|\,dxdt + \int_{\Omega_T\cap\{|u_n|\le 1\}} |h(u_n)|\,dxdt \\ &\le \int_{\Omega_T} h(u_n)u_n dxdt + \sup_{|s|\le 1}|h(s)||\Omega_T| \\ &= \int_{\Omega_T} f(u_n)u_n dxdt + \nu\int_{\Omega_T} |u_n|^2\,dxdt + |f(0)|\int_{\Omega_T} |u_n|\,dxdt \\ &+ \sup_{|s|\le 1}|h(s)||\Omega_T| \\ &\le C_3.\end{aligned}$$

This means that $\{h(u_n)\}$ is bounded in $L^1(\Omega_T)$, and so is $\{f(u_n)\}$. We rewrite the first equation of (1.1) in $L^q(0,T;W^{-1,q}(\Omega))+L^1(\Omega_T)$ as

$$u_{nt} = g + \operatorname{div}(a(\|\nabla u_n\|^p_{L^p(\Omega)})|\nabla u_n|^{p-2}\nabla u_n) - f(u_n). \tag{2.8}$$

Therefore, $\{u_{nt}\}$ is bounded in

$L^q(0,T;W^{-1,q}(\Omega))+L^1(\Omega_T)\subset L^1(0,T;W^{-1,q}(\Omega)+L^1(\Omega))$.

Since $W_0^{1,p}(\Omega)\cap L^\infty(\Omega)\subset\subset L^2(\Omega)\subset W^{-1,q}(\Omega)+L^1(\Omega)$, by the Aubin-Lions lemma, we see that $\{u_n\}$ is compact in $L^2(0,T;L^2(\Omega))$. Therefore, there is an a.e. convergent subsequence in Ω_T. Applying a diagonalization procedure and using Lemma 1.3 in [19, p. 12], we obtain (up to a subsequence) that

$u_n \rightharpoonup u$ in $L^p(0,T;W_0^{1,p}(\Omega))$,

$u_n \to u$ in $L^2(0,T;L^2(\Omega))$,

$u_{nt} \rightharpoonup u_t$ in $L^q(0,T;W^{-1,q}(\Omega))+L^1(\Omega_T)$,

$u_n(T)\to u(T)$ in $L^2(\Omega)$,

and

$$-\operatorname{div}(a\left(\|\nabla u_n\|^p_{L^p(\Omega)}\right)|\nabla u_n|^{p-2}\nabla u_n) \rightharpoonup -\chi \text{ in } L^q(0,T;W^{-1,q}(\Omega)). \tag{2.9}$$

We now pass to the limit in the nonlinear term. From (1.5) we see that $h(\cdot)$ is a strictly increasing function. Moreover, using (2.7) we have

$$\begin{aligned}\int_{\Omega_T} h(u_n(t))u_n(t)dxdt &\le \frac{1}{2}\|u_n(0)\|^2_{L^2(\Omega)} + TC(m,p,q,\lambda_1,|\Omega|)\|g\|^q_{L^2(\Omega)} \\ &+\frac{|f(0)|^2}{2}|\Omega|T + (\frac{1}{2}+\nu)\left(\|u_n(0)\|^2_{L^2(\Omega)} + C_2T\right).\end{aligned}$$

Since $u_n\to u$ strongly in $L^2(0,T;L^2(\Omega))$, then up to a subsequence, we have $u_n\to u$ a.e. in Ω_T. Applying Lemma 6.1 in [16], we obtain that $h(u)\in L^1(\Omega_T)$ and for all test function $\varphi\in C_0^\infty([0,T];W_0^{1,p}(\Omega)\cap L^\infty(\Omega))$,

$$\int_{\Omega_T} h(u_n)\varphi dxdt \to \int_{\Omega_T} h(u)\varphi dxdt \text{ as } n \to \infty.$$

Hence, $f(u) \in L^1(\Omega_T)$ and for all $\varphi \in C_0^\infty([0,T]; W_0^{1,p}(\Omega) \cap L^\infty(\Omega))$,

$$\int_{\Omega_T} f(u_n)\varphi dxdt \to \int_{\Omega_T} f(u)\varphi dxdt \text{ as } n \to \infty.$$

Now, passing to the limit in (2.8), one has in the distribution sense

$$u_t - \chi + f(u) = g. \tag{2.10}$$

It remains to prove that $\chi = Au$. To do this, integrating (2.6) from 0 to T we obtain

$$\int_0^T a\left(\| \nabla u_n \|_{L^p(\Omega)}^p\right) \int_\Omega |\nabla u_n|^p \, dxdt = \int_{\Omega_T} g u_n dxdt - \int_{\Omega_T} f(u_n) u_n dxdt + \frac{\| u_n(0) \|_{L^2(\Omega)}^2}{2} - \frac{\| u_n(T) \|_{L^2(\Omega)}^2}{2}.$$

Since $\lim\limits_{n\to\infty} \| u_n(T) \|_{L^2(\Omega)}^2 = \| u(T) \|_{L^2(\Omega)}^2$ and $\lim\limits_{n\to\infty} \| u_n(0) \|_{L^2(\Omega)}^2 = \| u_0 \|_{L^2(\Omega)}^2$, we deduce that

$$\lim_{n\to\infty} \int_0^T a\left(\| \nabla u_n \|_{L^p(\Omega)}^p\right) \int_\Omega |\nabla u_n|^p \, dxdt = \int_{\Omega_T} g u dxdt - \int_{\Omega_T} f(u) u dxdt + \frac{\| u_0 \|_{L^2(\Omega)}^2}{2} - \frac{\| u(T) \|_{L^2(\Omega)}^2}{2}. \tag{2.11}$$

Going back to (2.1), we have

$$\int_{\Omega_T} \left(a\left(\| \nabla u_n \|_{L^p(\Omega)}^p\right) |\nabla u_n|^{p-2} \nabla u_n - a\left(\| \nabla v \|_{L^p(\Omega)}^p\right) |\nabla v|^{p-2} \nabla v \right) \cdot \nabla(u_n - v) dxdt \geq 0$$

for all $v \in L^p(0,T; W_0^{1,p}(\Omega))$. Thus, taking limit leads to

$$\lim_{n\to\infty} \int_0^T a\left(\| \nabla u_n \|_{L^p(\Omega)}^p\right) \int_\Omega |\nabla u_n|^p \, dxdt + \int_0^T \langle \chi, v \rangle dt - \int_{\Omega_T} a\left(\| \nabla v \|_{L^p(\Omega)}^p\right) |\nabla v|^{p-2} \nabla v \cdot \nabla(u - v) dxdt \geq 0.$$

Putting this with (2.11), we have

$$\int_{\Omega_T} g u dxdt - \int_{\Omega_T} f(u) u dxdt + \frac{\| u_0 \|_{L^2(\Omega)}^2}{2} - \frac{\| u(T) \|_{L^2(\Omega)}^2}{2} + \int_0^T \langle \chi, v \rangle dt - \int_{\Omega_T} a\left(\| \nabla v \|_{L^p(\Omega)}^p\right) |\nabla v|^{p-2} \nabla v \cdot \nabla(u - v) dxdt \geq 0. \tag{2.12}$$

We see that $f(u) \in L^1(\Omega_T)$ and u does not belong to $W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$. Therefore, u cannot be chosen as a test function in (2.10). We will use some ideas in [17]. Let $B_k : \mathbb{R} \to \mathbb{R}$ be the truncated function defined by

$$B_k(s) = \begin{cases} k & \text{if } s > k, \\ s & \text{if } |s| \leq k, \\ -k & \text{if } s < -k. \end{cases}$$

We construct the following Nemytskii mapping

$$\hat{B}_k : W_0^{1,p}(\Omega)\cap L^\infty(\Omega) \to W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$$
$$v \mapsto \hat{B}_k(v)(x) = B_k(v(x)).$$

It follows from Lemma 2.3 in [17] that $\| \hat{B}_k(v) - v \|_{W_0^{1,p}(\Omega)\cap L^\infty(\Omega)} \to 0$ as $k \to \infty$. We now can test (2.10) by $\hat{B}_k(u)$. Multiplying (2.10) by $\hat{B}_k(u)$, then integrating from ε to T, we have

$$-\int_\varepsilon^T \langle \chi, \hat{B}_k(u)\rangle dt = \int_\varepsilon^T \int_\Omega g\hat{B}_k(u)dxdt - \int_\varepsilon^T \int_\Omega h(u)\hat{B}_k(u)dxdt$$
$$+\int_\varepsilon^T \int_\Omega (f(0)-\nu u)\hat{B}_k(u)dxdt + \int_\Omega u(\varepsilon)\hat{B}_k(u)(\varepsilon)dx$$
$$-\int_\Omega u(T)\hat{B}_k(u)(T)dx + \frac{1}{2}\| \hat{B}_k(u)(T)\|^2_{L^2(\Omega)} - \frac{1}{2}\| \hat{B}_k(u)(\varepsilon)\|^2_{L^2(\Omega)}.$$

Passing to the limit as $k\to\infty$ we have

$$-\int_\varepsilon^T \langle \chi, u\rangle dt = \int_\varepsilon^T \int_\Omega gudxdt - \lim_{k\to\infty}\int_\varepsilon^T \int_\Omega h(u)\hat{B}_k(u)dxdt$$

$$+\int_\varepsilon^T \int_\Omega (f(0)-\nu u)udxdt + \frac{1}{2}\| u(\varepsilon)\|^2_{L^2(\Omega)} - \frac{1}{2}\| u(T)\|^2_{L^2(\Omega)}, \tag{2.13}$$

where due to the nondecreasing of $\{h(u)\hat{B}_k(u)\}_{k=1}^\infty$ and $\hat{B}_k(u)\to u$ in $C([0,T];L^2(\Omega))$, it follows from the monotone convergence theorem that

$$\lim_{k\to\infty}\int_\varepsilon^T \int_\Omega h(u)\hat{B}_k(u)dxdt = \int_\varepsilon^T \int_\Omega h(u)udxdt.$$

We deduce from (2.13) by passing to the limit as $\varepsilon\to 0$ that

$$-\int_0^T \langle \chi, u\rangle dt = \int_{\Omega_T} gudxdt - \int_{\Omega_T} f(u)udxdt + \frac{\| u_0\|^2_{L^2(\Omega)}}{2} - \frac{\| u(T)\|^2_{L^2(\Omega)}}{2}. \tag{2.14}$$

In view of (2.12) and (2.14), we have

$$\int_0^T \langle \chi - \operatorname{div}(a\left(\| \nabla v\|^p_{L^p(\Omega)}\right)|\nabla v|^{p-2}\nabla v), u-v\rangle dt \le 0,\ \forall v\in L^p(0,T;W_0^{1,p}(\Omega)).$$

Choosing $v = u-\delta\varphi$, we deduce that

$$\int_0^T \langle \chi - \operatorname{div}(a(\|\nabla(u-\delta\varphi)\|^p_{L^p(\Omega)})|\nabla(u-\delta\varphi)|^{p-2}\nabla(u-\delta\varphi)), \varphi\rangle dt \le 0, \quad \text{if } \delta>0,$$

$$\int_0^T \langle \chi - \operatorname{div}(a(\|\nabla(u-\delta\varphi)\|^p_{L^p(\Omega)})|\nabla(u-\delta\varphi)|^{p-2}\nabla(u-\delta\varphi)), \varphi\rangle dt \ge 0, \quad \text{if } \delta<0,$$

for all $\varphi\in L^p(0,T;W_0^{1,p}(\Omega))$. Letting $\delta\to 0$, we get

$$\int_0^T \langle \chi - \operatorname{div}(a\left(\| \nabla u\|^p_{L^p(\Omega)}\right)|\nabla u|^{p-2}\nabla u), \varphi\rangle dt = 0,\ \forall \varphi\in L^p(0,T;W_0^{1,p}(\Omega)).$$

This implies that $\chi = \operatorname{div}\left(a\left(\|\nabla u\|_{L^p(\Omega)}^p\right)|\nabla u|^{p-2}\nabla u\right)$ in $L^q(0,T;W^{-1,q}(\Omega))$, which completes the proof of existence.

ii) Uniqueness and continuous dependence on the initial data

Let u, v be two weak solutions of (1.1) with initial data u_0, $v_0 \in L^2(\Omega)$, respectively. Then $w = u - v$ satisfies

$$\begin{cases} w_t - \operatorname{div}(a\left(\|\nabla u\|_{L^p(\Omega)}^p\right)|\nabla u|^{p-2}\nabla u) & +\operatorname{div}(a\left(\|\nabla v\|_{L^p(\Omega)}^p\right)|\nabla v|^{p-2}\nabla v) \\ & +f(u)-f(v)=0, \\ w(0) = u_0 - v_0. & \end{cases} \tag{2.15}$$

Multiplying the first equation in (2.15) by $\hat{B}_k(w)$, then integrating from ε to t, we obtain

$$\begin{aligned} &\int_\varepsilon^t\int_\Omega \frac{d}{ds}(w(s)\hat{B}_k(w)(s))dxds - \int_\varepsilon^t\int_\Omega w\frac{d}{ds}(\hat{B}_k(w)(s))dxds \\ &+\int_\varepsilon^t\int_\Omega \left(a\left(\|\nabla u\|_{L^p(\Omega)}^p\right)|\nabla u|^{p-2}\nabla u - a\left(\|\nabla v\|_{L^p(\Omega)}^p\right)|\nabla v|^{p-2}\nabla v\right)\cdot\nabla(\hat{B}_k(w)(s))dxds \\ &+\int_\varepsilon^t\int_\Omega (f(u)-f(v))\hat{B}_k(w)(s)dxds = 0. \end{aligned} \tag{2.16}$$

Since $w\frac{d}{dt}\hat{B}_k(w) = \frac{1}{2}\frac{d}{dt}(\hat{B}_k(w))^2$, we deduce from (2.1) and (1.5) by passing (2.16) to the limit as $k \to \infty$ and $\varepsilon \to 0$ that

$$\|w(t)\|_{L^2(\Omega)}^2 \le \|w(0)\|_{L^2(\Omega)}^2 + 2\ell\int_0^t \|w(s)\|_{L^2(\Omega)}^2\, ds.$$

An application of the Gronwall inequality of integral form leads to

$$\|w(t)\|_{L^2(\Omega)}^2 \le \|w(0)\|_{L^2(\Omega)}^2\, e^{2\ell t}, \text{ for all } t \in (0,T).$$

This implies the desired result.

3. Existence of global attractors

Theorem 2.1 allows us to construct a continuous (nonlinear) semigroup $S(t): L^2(\Omega) \to L^2(\Omega)$ associated to problem (1.1) as follows

$$S(t)u_0 := u(t),$$

where $u(t)$ is the unique global weak solution of (1.1) with the initial datum u_0.

3.1. Global attractor in $L^2(\Omega)$.

We first prove the following lemma.

Lemma 3.1. *The semigroup* $\{S(t)\}_{t\ge 0}$ *has a bounded absorbing set in* $L^2(\Omega)$.

Proof. From the first equation in (1.1), taking the inner product with u, we have

$$\frac{1}{2}\frac{d}{dt}\|u\|^2_{L^2(\Omega)}+a\left(\|\nabla u\|^p_{L^p(\Omega)}\right)\|u\|^p_{W_0^{1,p}(\Omega)}+\int_\Omega f(u)udx=\int_\Omega gudx. \tag{3.1}$$

In the case $p=2$, it follows from (1.2) and (1.4) that

$$\frac{1}{2}\frac{d}{dt}\|u\|^2_{L^2(\Omega)}+(m\lambda_1-\mu)\|u\|^2_{L^2(\Omega)}\le c_1|\Omega|+\int_\Omega gudx.$$

Since $m\lambda_1-\mu>0$, by the Young inequality, we obtain

$$\frac{d}{dt}\|u\|^2_{L^2(\Omega)}+(m\lambda_1-\mu)\|u\|^2_{L^2(\Omega)}\le 2c_1|\Omega|+\frac{1}{m\lambda_1-\mu}\|g\|^2_{L^2(\Omega)}.$$

In the case $p>2$, we deduce from (1.2) and (1.4) that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|u\|^2_{L^2(\Omega)}+\frac{1}{2}\|u\|^2_{L^2(\Omega)}+m\lambda_1\|u\|^p_{L^p(\Omega)}&\le\frac{1}{2}\|u\|^2_{L^2(\Omega)}-\int_\Omega f(u)udx+\int_\Omega gudx\\&\le(\mu+1)\|u\|^2_{L^2(\Omega)}+c_1|\Omega|+\frac{1}{2}\|g\|^2_{L^2(\Omega)}.\end{aligned}$$

Moreover, there exists a positive constant C_5 such that

$$-m\lambda_1|s|^p+(\mu+1)s^2\le C_5.$$

Thus, in both cases we have

$$\frac{d}{dt}\|u\|^2_{L^2(\Omega)}+\|u\|^2_{L^2(\Omega)}\le 2(C_5+c_1)|\Omega|+\|g\|^2_{L^2(\Omega)}.$$

Applying the Gronwall inequality, we get

$$\|u(t)\|^2_{L^2(\Omega)}\le\|u_0\|^2_{L^2(\Omega)}e^{-R_1t}+R_2(1-e^{-R_1t}), \tag{3.2}$$

where $R_1=m\lambda_1-\mu, R_2=2c_1|\Omega|+\dfrac{1}{m\lambda_1-\mu}\|g\|^2_{L^2(\Omega)}$ if $p=2$ and $R_1=1$, $R_2=2(C_5+c_1)|\Omega|+\|g\|^2_{L^2(\Omega)}$ if $p>2$. Therefore,

$$\|u(t)\|^2_{L^2(\Omega)}\le\rho_0 \tag{3.3}$$

for all $t\ge T_0=T_0(\|u_0\|^2_{L^2(\Omega)})$, where $\rho_0=2R_2$ is independent of u_0.

Lemma 3.2. *The semigroup $\{S(t)\}_{t\ge0}$ has a bounded absorbing set in $W_0^{1,p}(\Omega)$.*

Proof. Multiplying the first equation in (1.1) by $-\Delta_p u$ and integrating by parts, we obtain

$$\frac{1}{p}\frac{d}{dt}\|u\|^p_{W_0^{1,p}(\Omega)}+a\left(\|\nabla u\|^p_{L^p(\Omega)}\right)\|\Delta_p u\|^2_{L^2(\Omega)}=-\int_\Omega f'(u)|\nabla u|^p dx-\int_\Omega g\Delta_p udx.$$

Using (1.2), (1.4) and the Cauchy inequality, we deduce that

$$\frac{d}{dt}\|u\|^p_{W_0^{1,p}(\Omega)}\le\ell p\|u\|^p_{W_0^{1,p}(\Omega)}+\frac{p}{4m}\|g\|^2_{L^2(\Omega)}. \tag{3.4}$$

On the other hand, integrating (3.1) from t to $t+1$ and using (1.3) together with the Cauchy inequality, we have

$$\int_t^{t+1} a\left(\| \nabla u(s) \|^p_{L^p(\Omega)}\right) \| u(s) \|^p_{W_0^{1,p}(\Omega)} \, ds + \frac{1}{2} \| u(t+1) \|^2_{L^2(\Omega)}$$
$$\leq \left(\mu + \frac{1}{2}\right) \int_t^{t+1} \| u(s) \|^2_{L^2(\Omega)} \, ds + \frac{1}{2} \| u(t) \|^2_{L^2(\Omega)} + c_1 |\Omega| + \frac{1}{2} \| g \|^2_{L^2(\Omega)} .$$

In view of (3.2) and (1.2), we get the following estimate

$$\int_t^{t+1} \| u(s) \|^p_{W_0^{1,p}(\Omega)} \, ds \leq \frac{1}{m}\left((\mu + \frac{1}{2})\rho_0 + c_1 |\Omega| + \frac{1}{2} \| g \|^2_{L^2(\Omega)} \right), \tag{3.5}$$

for all $t \geq T_0$. As an application of the uniform Gronwall inequality, we deduce from (3.4) and (3.5) that

$$\| u(t) \|^p_{W_0^{1,p}(\Omega)} \leq \rho_1, \tag{3.6}$$

for all $t \geq T_1 = T_0 + 1$, where $\rho_1 = \left[\frac{2\mu+1}{2m}\rho_0 + c_1 |\Omega| + \frac{p+2}{4m} \| g \|^2_{L^2(\Omega)} \right] e^{\ell p}$.

The following theorem is a direct consequence of Lemma 3.2 and the compactness of the embedding $W_0^{1,p}(\Omega) \subset\subset L^2(\Omega)$.

Theorem 3.1. *Under the assumptions* **(H1) - (H3)**, *the semigroup* $\{S(t)\}_{t\geq 0}$ *generated by problem (1.1) has a compact global attractor* $\mathcal{A}_2$ *in* $L^2(\Omega)$.

3.2. Global attractor in $W_0^{1,p}(\Omega)$

In this subsection we will prove the existence of a global attractor in $W_0^{1,p}(\Omega)$ under the following additional assumption

(H1bis) a is a continuously differentiable and nondecreasing function satisfying **(H1)**.

We first define the following subset

$$\mathbb{B}_R = \left\{ u \in L^2(\Omega) : \ \| u \|_{W_0^{1,p}(\Omega)} + \| \Delta_p u \|_{L^2(\Omega)} \leq R \right\}.$$

We see that $\mathbb{B}_R$ is the subset of the domain of Δ_p acting on $L^2(\Omega)$. Moreover, it is is precompact in $W_0^{1,p}(\Omega)$ (see [16, Remark 4.3]). We have the following important lemma.

Lemma 3.3. *Under the assumptions* **(H1bis), (H2)** *and* **(H3).** *For $R > 0$ sufficiently large, $\mathbb{B}_R$ is an absorbing set for the semigroup $S(t)$ acting on $L^2(\Omega)$ (hence absorbing on $W_0^{1,p}(\Omega)$).*

Proof. It is enough to prove that the bounded absorbing set in Lemma 3.2 is absorbed into $\mathbb{B}_R$ for some $R > 0$. Indeed, we denote $v = u_t$. By differentiating the first equation in (1.1) in time, we obtain

$$v_t - \mathrm{div}\Big(a\big(\| \nabla u \|^p_{L^p(\Omega)}\big) | \nabla u |^{p-2} \nabla v\Big)$$
$$-(p-2)\mathrm{div}\Big(a\big(\| \nabla u \|^p_{L^p(\Omega)}\big) | \nabla u |^{p-4} (\nabla u \cdot \nabla v)\nabla u\Big)$$
$$-p\, \mathrm{div}\Big(a'\big(\| \nabla u \|^p_{L^p(\Omega)}\big) \int_\Omega |\nabla u |^{p-2} (\nabla u \cdot \nabla v)dx \,| \nabla u |^{p-2} \nabla u\Big) + f'(u)v = 0.$$

Taking the inner product of the above equality with v and using (1.4), one gets

$$\frac{1}{2}\| v \|^2_{L^2(\Omega)} + a\big(\| \nabla u \|^p_{L^p(\Omega)}\big)\int_\Omega |\nabla u |^{p-2} | \nabla v |^2 \, dx$$
$$+(p-2)a\big(\| \nabla u \|^p_{L^p(\Omega)}\big)\int_\Omega |\nabla u |^{p-4} (\nabla u \cdot \nabla v)^2 dx$$
$$+p\, a'\big(\| \nabla u \|^p_{L^p(\Omega)}\big)\Big(\int_\Omega |\nabla u |^{p-2} (\nabla u \cdot \nabla v)dx\Big)^2 \le \ell \| v \|^2_{L^2(\Omega)} .$$

By assumption **(H1bis)**, it follows from the last inequality that

$$\frac{d}{dt}\| v \|^2_{L^2(\Omega)} \le 2\ell \| v \|^2_{L^2(\Omega)} . \tag{3.7}$$

On the other hand, multiplying the first equation in \eqref{eq:1.1} by \$u_t\$, we get

$$\| u_t \|^2_{L^2(\Omega)} + \frac{1}{p} a\big(\| \nabla u \|^p_{L^p(\Omega)}\big)\frac{d}{dt}\| u \|^p_{W_0^{1,p}(\Omega)} + \int_\Omega f(u)u_t dx - \int_\Omega g u_t dx = 0.$$

We can rewrite this equality as follows

$$\| u_t \|^2_{L^2(\Omega)} + \frac{d}{dt}\Big[\frac{1}{p} a\big(\| \nabla u \|^p_{L^p(\Omega)}\big)\int_\Omega |\nabla u |^p \, dx + \int_\Omega F(u)dx - \int_\Omega gudx\Big]$$
$$= \frac{1}{p} a'\big(\| \nabla u \|^p_{L^p(\Omega)}\big)\| \nabla u \|^p_{L^p(\Omega)} \frac{d}{dt}\| \nabla u \|^p_{L^p(\Omega)} .$$

Setting $L = \sup\limits_{0\le s\le \rho_1} | a'(s) |$.

In view of **(H1bis)**, (3.4) and (3.6), we deduce that

$$\| u_t \|^2_{L^2(\Omega)} + \frac{d}{dt}\Big[\frac{1}{p} a\big(\| \nabla u \|^p_{L^p(\Omega)}\big)\| u \|^p_{W_0^{1,p}(\Omega)} + \int_\Omega F(u)dx - \int_\Omega gudx\Big] \tag{3.8}$$
$$\le L\rho_1\Big(\ell\rho_1 + \frac{1}{4m}\| g \|^2_{L^2(\Omega)}\Big).$$

On the other hand, integrating (3.1) from t to $t+1$ and using (3.3) leads to

$$\int_t^{t+1}\Big[a\big(\| \nabla u \|^p_{L^p(\Omega)}\big)\| u \|^p_{W_0^{1,p}(\Omega)} + \int_\Omega f(u)udx - \int_\Omega gudx\Big]ds \le \frac{\rho_0}{2},$$

for all $t \ge T_0$. It follows from (1.5) that

$$-\frac{\ell+1}{2}u^2 - \frac{| f(0)|^2}{2} \le F(u) \le f(u)u + \frac{\ell}{2}u^2, \text{ for all } u \in \mathbb{R}.$$

Hence, we have

$$\int_t^{t+1}\Big[\frac{1}{p}a\left(\|\nabla u\|_{L^p(\Omega)}^p\right)\|u\|_{W_0^{1,p}(\Omega)}^p+\int_\Omega F(u)dx-\int_\Omega gudx\Big]ds\le\frac{\ell+1}{2}\rho_0, \tag{3.9}$$

for all $t\ge T_0$. Using the uniform Gronwall inequality, it follows from (3.8) and (3.9) that

$$\frac{1}{p}a\left(\|\nabla u\|_{L^p(\Omega)}^p\right)\|u\|_{W_0^{1,p}(\Omega)}^p+\int_\Omega F(u)dx-\int_\Omega gudx\le\rho_2, \tag{3.10}$$

for all $t\ge T_2=T_1+1$, and $\rho_2=\frac{\ell+1}{2}\rho_0+L\rho_1\left(\ell\rho_1+\frac{1}{4m}\|g\|_{L^2(\Omega)}^2\right)$. Integrating (3.8) from t to $t+1$ and using (3.10), we infer that

$$\int_t^{t+1}\|u_t\|_{L^2(\Omega)}^2ds\le(\ell+1)\rho_0+3L\rho_1\left(\ell\rho_1+\frac{1}{4m}\|g\|_{L^2(\Omega)}^2\right),\text{ for all } t\ge T_2. \tag{3.11}$$

Using the uniform Gronwall inequality again, it follows from (3.7) and (3.11) that

$$\|u_t(t)\|_{L^2(\Omega)}^2\le\left[(\ell+1)\rho_0+3L\rho_1\left(\ell\rho_1+\frac{1}{4m}\|g\|_{L^2(\Omega)}^2\right)\right]e^{2\ell}, \tag{3.12}$$

for all $t\ge T_3=T_2+1$. On the other hand, multiplying the first equation in (1.1) by $-\Delta_p u$, using (1.4) and the Cauchy inequality, we obtain

$$\begin{aligned}a\left(\|\nabla u\|_{L^p(\Omega)}^p\right)\|\Delta_p u\|_{L^2(\Omega)}^2&=\int_\Omega u_t\Delta_p udx-\int_\Omega f'(u)|\nabla u|^p\,dx-\int_\Omega g\Delta_p udx\\&\le\ell\|u\|_{W_0^{1,p}(\Omega)}^p+\frac{m}{2}\|\Delta_p u\|_{L^2(\Omega)}^2+\frac{1}{m}\|u_t\|_{L^2(\Omega)}^2+\frac{1}{m}\|g\|_{L^2(\Omega)}^2.\end{aligned}$$

The following estimate is obtained from (1.2), (3.3), (3.6) and (3.12),

$$\|\Delta_p u(t)\|_{L^2(\Omega)}^2\le\frac{2\ell}{m}\rho_1+\frac{2}{m^2}\rho_0+\frac{2}{m^2}\|g\|_{L^2(\Omega)}^2,\text{ for all } t\ge T_3.$$

This combining with (3.6) implies the desired result.

By the similar arguments of Corollary 4.5 in [16], we get the following result.

Theorem 3.4. *Under the assumptions* **(H1bis), (H2)** *and* **(H3)**, *the semigroup* $\{S(t)\}_{t\ge0}$ *associated to problem (1.1) has a compact global attractor* $\mathcal{A}$ *in* $W_0^{1,p}(\Omega)$.

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