# ON THE $C_g$-ASYMPTOTIC EQUIVALENCE OF DIFFERENTIAL EQUATIONS WITH MAXIMA

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**Abstract:** *The aim of this paper is to investigate the $C_g$-asymptotic equivalence of differential equations with maximum by applying the Banach's fixed point theorem in an appropriately weighted function space. In some contexts, this paper generalizes the results of D. Otrocol [8].*

**Keywords:** *Differential equations with maximum, asymptotic equivalence.*

## 1. Introduction

Differential equations with "maxima" are a special type of differential equations that contain the maximum of the unknown function over a previous interval(s). Such equations adequately model real world processes whose present state significantly depends on the maximum value of the state on a past time interval. For example, in the theory of automatic control of various technical systems, it often occurs that the law of regulation depends on the maximum values of some regulated state parameters over certain time intervals. In [6], E. P. Popov considered the system for regulating the voltage of a generator of constant current. The object of the experiment was a generator of constant current with parallel simulation and the regulated quantity was the voltage at the source electric current. The equation describing the work of the regulator involves the maximum of the unknown function and it has the form

$$T_0 u'(t) = -u(t) - q \max_{s\in[t-h,t]} u(s) + f(t), \tag{1.1}$$

where $T_0$ and $q$ are constants characterizing the object, $u(t)$ is the regulated voltage and $f(t)$ is the effect of the perturbation that appears associated to the change of voltage.

Recently, the interest in differential equations with "maxima" has increased exponentially. Let us mention, for instance, iteration method for approximating solutions of nonlinear differential equations with maxima [1] [2] [3]; qualitative properties of solutions [4] [5] [9] [10], etc.

Let $\mathbb{R}^n$ be the Euclidian $n$-space. For $u=(u_1,u_2,\cdots,u_n)^T \in \mathbb{R}^n$, let $\| u \| := \max\{|u_1|,\cdots,|u_n|\}$ be the norm of $u$. For a matrix $A \in M_{n\times n}(\mathbb{R}), A=(a_{ij})$, we define the norm $|A|$ of $A$ by $|A| = \max\limits_{1\leqslant i\leqslant n} \sum\limits_{j=1}^{n} |a_{ij}|$.

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In this paper, we consider the following differential system

$$\dot{x}(t) = A(t)x(t), \quad t \geqslant t_0 \tag{1.2}$$

and the perturbed one with maxima of the form

$$\dot{y}(t) = A(t)y(t) + f\left(t, y(t), \max_{\xi \in [t_0, t]} y(\xi)\right), \quad t \geqslant t_0. \tag{1.3}$$

More precisely, the following Cauchy problem

$$y'(t) = A(t)y(t) + f\left(t, y(t), \max_{t_0 \le \xi \le t} y(\xi)\right), \ t \in [t_0, \infty) \tag{1.4}$$

$$y(t_0) = y_0. \tag{1.5}$$

**Definition 1.1** ([8]). The Equations (1.2) and (1.3) are asymptotically equivalent if for every solution $x$ of (1.2), there is a solution $y$ of (1.3) such that

$$\lim_{t \to \infty} \| x(t) - y(t) \| = 0, \tag{1.6}$$

and conversely, for each solution $y$ of (1.3) there exists a solution $x$ of (1.2) such that (1.6) holds. In [8], Otrocol proved the asymptotical equivalence of (1.2) and (1.3). Impressed by more general concept of this notion which introduced in [7] by Olaru called " $C_g$ - asymptotic equivalence", in this paper we extend the result of Otrocol to the case of $C_g$ - asymptotic equivalence. To do this, we make the following assumption.

**Assumption 1.2.** There exists $L_f : [a, \infty) \to \mathbb{R}_+$ such that

$$\| f(t, u_1, u_2) - f(t, v_1, v_2) \| \le L_f(t) \max\{| u_1 - v_1 |, | u_2 - v_2 |\}, \ \forall t \in [t_0, \infty), \ u_i, v_i \in \mathbb{R}^n, i = 1,2;$$

## 2. The existence of solution in weighted space

Now, for $g$ is a given continuous and positive function defined on $[t_0, \infty)$, we define

$$BC_g = \left\{ u \in C\left([t_0, \infty), \mathbb{R}^n\right) : \sup_{t \geqslant t_0} g^{-1}(t) \| u(t) \| < \infty \right\}.$$

the Banach space endowed with the norm

$$\| x \|_{BC_g} = \sup_{t \geqslant t_0} | g^{-1}(t)x(t) |, \ \forall x \in BC_g.$$

**Theorem 2.1.** Let $X(t)$ be a fundamental matrix of the system (1.2). Suppose that

a) the Assumption 1.2 holds;

b) there exists a constant $K > 0$ such that

$$\| g^{-1}(t)X(t)X^{-1}(s)g(s) \| \le K, \ t_0 \le s \le t < \infty;$$

$c) \int_{t_0}^{\infty} \| g^{-1}(s) f(s, 0, 0) \| \, ds < \infty$;

$d) K \int_{t_0}^{\infty} L_f(s) ds < 1$.

Then the problem (1.4) - (1.5) has a unique solution in $BC_g([t_0,\infty),\mathbb{R}^n)$.

*Proof.* For $y \in BC_g([t_0,\infty),\mathbb{R}^n)$ we put

$$(Fy)(t) := X(t)X^{-1}(t_0)y_0 + \int_{t_0}^{t} X(t)X^{-1}(s)f\left(s, y(s), \max_{t_0\leqslant\xi\leqslant s} y(\xi)\right)ds, \ \ \forall t \ge t_0.$$

We have

$$\begin{aligned}
|(Fy)(t)| &\le Mg(t)\,|g^{-1}(t)X(t)X^{-1}(y_0)y_0| \\
&+ g(t)\int_{t_0}^{t} \| g^{-1}(t)X(t)X^{-1}(s)g(s)\| g^{-1}(s)\left| f\left(s, y(s), \max_{t_0\le\xi\le s} y(s)\right) - f(s,0,0)\right| ds \\
&+ g(t)\int_{t_0}^{t} \| g^{-1}(t)X(t)X^{-1}(s)g(s)\| g^{-1}(s)\left|f(s,0,0)\right| ds \\
&\le g(t)Ky_0 + g(t)K\int_{t_0}^{t} L_f(s)g^{-1}(s)\max\left\{|y(s)|, \left|\max_{t_0\le\xi\le s} y(\xi)\right|\right\} \\
&+ g(t)K\int_{t_0}^{t} |g^{-1}(s)f(s,0,0)|\,ds \\
&\le g(t)Ky_0 + g(t)K\,\|y\|_{BC_g}\int_{t_0}^{\infty} L_f(s)ds + g(t)K\int_{t_0}^{\infty}|g^{-1}(s)f(s,0,0)|\,ds,
\end{aligned}$$

which implies

$$\sup_{t\ge t_0}\frac{|(Fy)(t)|}{g(t)} \le K\left( y_0 + \|y\|_{BC_g}\int_{t_0}^{\infty} L_f(s)ds + \int_{t_0}^{\infty}|g^{-1}(s)f(s,0,0)|\,ds \right) < \infty.$$

Thus, $F : BC_g([t_0,\infty),\mathbb{R}^n) \to BC_g([t_0,\infty),\mathbb{R}^n)$.

Furthermore, for any $y_1, y_2 \in BC_g([t_0,\infty),\mathbb{R}^n)$ one has

$$\begin{aligned}
&|(Fy_1)(t) - (Fy_2)(t)| \\
&= \Bigg| X(t)X^{-1}(t_0)y_0 + \int_{t_0}^{t} X(t)X^{-1}(s)f\left(s, y_1(s), \max_{t_0\le\xi\le s} y_1(\xi)\right)ds \\
&\quad - X(t)X^{-1}(a)y_0 - \int_{t_0}^{t} X(t)X^{-1}(s)f\left(s, y_2(s), \max_{t_0\le\xi\le s} y_2(\xi)\right)ds \Bigg| \\
&\le g(t)\int_{t_0}^{t} \left\| g^{-1}(t)X(t)X^{-1}(s)g(s)\right\| L_f(s)g^{-1}(s)\max\left\{|y_1(s)-y_2(s)|, \left|\max_{t_0\le\xi\le s} y_1(\xi) - \max_{t_0\le\xi\le s} y_2(\xi)\right|\right\} \\
&\le g(t)K\int_{t_0}^{t} L_f(s)g^{-1}(s)\max\left\{|y_1(s)-y_2(s)|, \max_{t_0\le\xi\le s}|y_1(\xi)-y_2(\xi)\right\} ds \\
&\le g(t)K\int_{t_0}^{t} L_f(s)ds\,\|y_1, -y_2\|_{BC_g}.
\end{aligned}$$

Here, we conclude that

$$\| Ty_1 - Ty_2 \|_{BC_g} \le K \int_{t_0}^{\infty} L_f(s)ds \cdot \| y_1, -y_2 \|_{BC_g}, \ \forall \ y_1, y_2 \in BC_g([t_0, \infty), \mathbb{R}^n).$$

By the condition d), $T$ is a contraction mapping on $BC_g([t_0,\infty),\mathbb{R}^n)$. Thus, there exists a unique solution to (1.3) in $BC_g([t_0,\infty),\mathbb{R}^n)$ which is a fixed point of $T$.

## 3. $C_g$ - Asymptotic equivalence

**Definition 3.1** ([7]). Let $g(t)$ be a positive continuous function on $[t_0,\infty)$. We say that Equation (1.2) and (1.3) (or the system (1.4) - (1.5)) are $C_g$ - equivalent on $[t_0,\infty)$ if and only if to each solution $x(\cdot,t_0)$ of (1.2) there exists a solution $y(\cdot,t_0)$ of (1.3) such that

$$\lim_{t\to\infty} \frac{| x(t,t_0) - y(t,t_0) |}{g(t)} = 0 \tag{3.1}$$

and conversely, for each solution $y$ of (1.3) there exists a solution $x$ of (1.2) such that (3.1) holds.

**Theorem 3.2.** Let $X(t)$ be a fundamental matrix of the system (1.2). We suppose that

a) the Assumption 1.2 holds;

b) $\int_{t_0}^{\infty} g^{-1}(s) \| f(s,0,0) \| ds < \infty$;

c) there exists two projectors $P_1, P_2$ of $\mathbb{R}^n$ and a constant $K > 0$ such that

$$\| g(t)X(t)P_1X^{-1}(s)g^{-1}(s) \| \le K, \ \text{for } t_0 \le s \le t,$$
$$\| g(t)X(t)P_2X^{-1}(s)g^{-1}(s) \| \le K, \ \text{for } t_0 \le t \le s,$$
$$\lim_{t\to\infty} g^{-1}(t)X(t)P_1 = 0;$$

d) $2K\int_{t_0}^{\infty} L_f(s)ds < 1$.

Then (1.2) and (1.3) are $C_g$ - asymptotically equivalent.

*Proof.* Let $x$ be a $BC_g$-solution of (1.2).

Corresponding to $x$ we consider the operator

$$Ty(t) = x(t) + \int_{t_0}^{t} X(t)P_1X^{-1}(s) f\left(s, y(s), \max_{t_0\le\xi\le s} y(\xi)\right) ds -$$
$$- \int_{t}^{\infty} X(t)P_2X^{-1}(s) f\left(s, y(s), \max_{t_0\le\xi\le s} y(\xi)\right) ds.$$

We show that $BC_g\left([t_0,\infty),\mathbb{R}^n\right)$ is invariant for the operator $T$.

If $y \in BC_g\left([t_0,\infty),\mathbb{R}^n\right)$ then

$$\left|f\left(t,y(t),\max_{t_0\le\xi\le t} y(\xi)\right)\right| \le \left|f\left(t,y(t),\max_{t_0\le\xi\le t} y(\xi)\right)-f(t,0,0)\right|+|f(t,0,0)|$$

$$\le L_f(t)\max\left(|y(t),\left|\max_{\xi\in[t_0,t]} y(\xi)\right|\right)+|f(t,0,0)|$$

$$\le L_f(t)\|y\|+|f(t,0,0)|.$$

Let $x$ be a $BC_g$ - solution of (1.2) and $y \in BC_g\left([t_0,\infty),\mathbb{R}^n\right)$. Then

$$|Ty(t)|\le Mg(t)+g(t)\int_{t_0}^{t}\left|g^{-1}(t)X(t)P_1X^{-1}(s)g(s)g^{-1}(s)f\left(t,y(t),\max_{t_0\le\xi\le t} y(\xi)\right)\right|ds$$

$$+g(t)\int_{t}^{\infty}\left|g^{-1}(t)X(t)P_2X^{-1}(s)g(s)g^{-1}(s)f\left(s,y(s),\max_{t_0\le\xi\le s} y(\xi)\right)\right|ds$$

$$\le Mg(t)+g(t)K\|y\|_{BC_g}\int_{t_0}^{t}L_f(s)ds+g(t)K\int_{t_0}^{t}|g^{-1}(s)f(s,0,0)|ds$$

$$+g(t)K\|y\|_{BC_g}\int_{t}^{\infty}L_f(s)ds+g(t)K\int_{t_0}^{\infty}g^{-1}(s)|f(s,0,0)|ds.$$

So

$$\|Ty\|_{BC_g}\le M+K\|y\|_{BC_g}\int_{t_0}^{\infty}L_f(s)ds+K\int_{t_0}^{\infty}g^{-1}(s)|f(s,0,0)|ds<\infty.$$

Now we prove that $T$ is a contraction on $BC_g\left([t_0,\infty),\mathbb{R}^n\right)$

$$|Ty_1(t)-Ty_2(t)|$$

$$\le g(t)\int_{t_0}^{t}|g^{-1}(t)X(t)P_1X^{-1}(s)g(s)|g^{-1}(s)\left|f\left(s,y_1(s),\max_{t_0\le\xi\le s} y_1(\xi)\right)-f\left(s,y_2(s),\max_{t_0\le\xi\le s} y_2(\xi)\right)\right|$$

$$+g(t)\int_{t}^{\infty}|g^{-1}(t)X(t)P_2X^{-1}(s)g(s)|g^{-1}(s)\left|f\left(s,y_1(s),\max_{t_0\le\xi\le s} y_1(\xi)\right)-f\left(s,y_2(s),\max_{t_0\le\xi\le s} y_2(\xi)\right)\right|$$

$$\le g(t)K\int_{t_0}^{t}L_f(s)g^{-1}(s)\max\left(|y_1(s)-y_2(s),\left|\max_{t_0\le\xi\le s} y_1(s)-\max_{t_0\le\xi\le s} y_2(s)\right|\right)ds$$

$$+g(t)K\int_{t}^{\infty}L_f(s)g^{-1}(s)\max\left(|y_1(s)-y_2(s),\left|\max_{t_0\le\xi\le s} y_1(s)-\max_{t_0\le\xi\le s} y_2(s)\right|\right)ds$$

$$\le g(t)K\|y_1-y_2\|_{BC_g}\int_{t_0}^{t}L_f(s)ds+g(t)K\|y_1-y_2\|_{BC_g}\int_{t}^{\infty}L_f(s)ds,\ \forall y_1,y_2\in BC_g\left([t_0,\infty),\mathbb{R}^n\right).$$

Therefore,

$$\|Ty_1-Ty_2\|_{BC_g}\le\left(2K\int_{t_0}^{\infty}L_f(s)ds\right)\|y_1-y_2\|_{BC_g}\quad\forall y_1,y_2\in BC_g\left([t_0,\infty),\mathbb{R}^n\right).$$

By Banach's fixed point theorem, there exists a unique solution of (1.3). Let $y$ be a solution of (1.3) corresponding to $x$. Then

$$
\begin{aligned}
&| x(t) - y(t) | \\
&\leq g(t)\int_{t_0}^{t}\left|g^{-1}(t)X(t)P_1X^{-1}g(s)g^{-1}(s)f\left(s, y_1(s), \max_{t_0\leq\xi\leq s} y_1(\xi)\right)\right|ds \\
&\quad + g(t)\int_{t}^{\infty}\left|g^{-1}(t)X(t)P_2X^{-1}(s)g(s)g^{-1}(s)f\left(s, y_1(s), \max_{t_0\leq\xi\leq s} y_1(\xi)\right)\right|ds \\
&\leq g(t)\int_{t_0}^{t_1}\left|g^{-1}(t)X(t)P_1X^{-1}g(s)g^{-1}(s)f\left(s, y_1(s), \max_{t_0\leq\xi\leq s} y_1(\xi)\right)\right|ds \\
&\quad + g(t)\int_{t_1}^{t}\left|g^{-1}(t)X(t)P_1X^{-1}(s)g(s)g^{-1}(s)f\left(s, y_1(s), \max_{t_0\leq\xi\leq s} y_1(\xi)\right)\right|ds \\
&\quad + g(t)\int_{t}^{\infty}\left|g^{-1}(t)X(t)P_2X^{-1}(s)g(s)g^{-1}(s)f\left(s, y_1(s), \max_{t_0\leq\xi\leq s} y_1(\xi)\right)\right|ds.
\end{aligned}
$$

Hence

$$\| x - y \|_{BC_g} | X(t)P_1 | \int_{t_0}^{t_1} | X^{-1}(s) | \left|f\left(s, y(s), \max_{t_0\leq\xi\leq s} y(\xi)\right)\right|ds + K\int_{t_1}^{\infty}\left|f\left(s, y(s), \max_{t_0\leq\xi\leq s} y(\xi)\right)\right|ds .$$

We fix $\varepsilon > 0$ and determine $t_1 \geq t_0$ such that the second integral in the right-hand side of the above inequality is less than $\varepsilon / 2$. With $t_1$ fixed, from hypothesis c), we have that the first term on the right-hand side of the above inequality tends to zero as $x$ tends to infinity. We may therefore conclude that $\lim_{t\to\infty}\frac{\| x(t) - y(t) \|}{g(t)} = 0.$

Similarly, we take a bounded solution $y$ of (1.3) and we observe that

$$
\begin{aligned}
x(t) = y(t) - &\int_{t_0}^{t} X(t)P_1X^{-1}(s)f\left(s, y(s), \max_{t_0\leq\xi\leq s} y(\xi)\right)ds \\
&+ \int_{t}^{\infty} X(t)P_2X^{-1}(s)f\left(s, y(s), \max_{t_0\leq\xi\leq s} y(\xi)\right)ds
\end{aligned}
$$

satisfies (1.2) and (1.6) holds.

**Remark 3.3.** Note that, when $g \equiv 1$ we obtain the result which is presented by Otrocol [8] as mentioned before.

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