FRÉCHET SINGULAR SUBDIFFERENTIALS OF THE MINIMAL TIME FUNCTION ASSOCIATED WITH A COLLECTION OF SETS

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Abstract: *In this paper, we present the formulas for computing the Fréchet singular subdifferentials of the minimal time function associated with a collection of subsets in normed spaces.*

Keywords: *Minimal time function, Fréchet singular subdifferentials, Normed spaces.*

1. Introduction and preliminaries

Let X be a normed space, F and Ω be two nonempty subsets of X. The minimal time function with the constant dynamics F and the target set Ω is defined by

$$T_{\Omega}^{F}(x) := \inf\{t \geq 0 : (x+tF) \cap \Omega \neq \emptyset\}, \quad x \in X. \tag{1.1}$$

The minimal time function T_{Ω}^{F} plays an important role in variational analysis since its covers three crucial functions in variational analysis: the distance function, the Minkowski function and the indicator function. Variational analysis and subdifferentials of the minimal time function with a convex dynamics containing the origin in its interior, in Hilbert spaces were first investigated by Colombo and Wolenski in [5] [6]. Later, the function has been studied extensively by many researchers; see, e.g., [2] [3] [4] [7] [8] [9] [12] [13] [16] [19] [21]. He [8] studied subdifferentials of the minimal time function in Banach spaces. These results were extended to the setting of normed spaces by Jiang and He in [9] and then improved by Mordukhovich and Nam in [12] [13]. Bounkhel investigated subdifferential calculus of the minimal time function in Hausdorff topological vector spaces in [2] [3]. Applications of variational analysis and generalized differentiations of the minimal time function to generalized location problems were presented in [13] [14] [15] [16] [17] [18] [20] and references therein.

The notion of the minimal time function associated with a collection of set was recently introduced in [11]. This new function contains the classical minimal time function as a special case. More precisely, let m be a positive integer and let $\mathcal{U} = \{U_1, \cdots, U_m\}$ be a collection of m nonempty subsets $U_1, \cdots, U_m$ of X and Ω a nonempty subset of X. The minimal time function associated with the collection $\mathcal{U}$ to the set Ω is defined as: for $x \in X$.

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$$T_{\mathcal{U},\Omega}(x) := \inf\left\{t_1 + \cdots + t_m : t_1, \cdots, t_m \geq 0 \text{ and} \left(x + t_1U_1 + \cdots + t_mU_m\right) \cap \Omega \neq \varnothing\right\}.$$

It is obvious that if $U_1 = F$ and $U_2 = \cdots = U_m = \{0\}$, then $T_{\mathcal{U},\Omega}$ becomes the usual minimal time function T_Ω^F defined in (1.1). Let $x \in X$. From the definition of the minimal time function $T_{\mathcal{U},\Omega}$, we see that if $T_{\mathcal{U},\Omega}(x) < \infty$, then it is the smallest time to steer x to the target Ω using at most one direction in each set $U_1, \cdots, U_m$. It means that x may be steered to the target Ω in a "zigzag" path. This contrasts with the case of the classical minimal time function as points are steered to the target along a straight path. It turns out that the new type of minimal time function is more flexible than the classical one and it can be used to model problems that the classical one cannot. By careful adaptation of existing results for the classical minimal time function, in [11], we present various basic properties of the new minimal time function. These properties (which include, among others, lower semicontinuity, Lipschitz continuity, convexity, principle of optimality and subdifferential calculus) were then utilized to study a location problem. The aim of this paper is to continue investigating the minimal time function associated with a collection of sets by proving the formulas for computing Fréchet singular subdifferentials of the function.

We now recall some basic concepts of nonsmooth analysis [1] [10]. Let X be normed space and X^* be the topological dual space of X. We denote by $\|\cdot\|$ the norm in X and by $\langle\cdot,\cdot\rangle$ the dual pair between X and X^*. We also denote by $\|\cdot\|$ the dual norm in X^*. Denote by $B(x,r)$ the open ball of radius $r > 0$ centered at x and $\mathbb{B} = B(0,1)$.

Let $S \subset X$ be a closed set and let $x \in S$. The Fréchet normal cone to S at x, written $\hat{N}_S(x)$, is the set

$$\hat{N}_S(x) := \left\{\zeta \in X^* : \limsup_{S \ni y \to x} \frac{\langle \zeta, y - x\rangle}{\| y - x \|} \leq 0\right\}.$$

In other words, $\zeta \in \hat{N}_S(x)$ if and only if for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\langle \zeta, y - x\rangle \leq \varepsilon \| y - x \|, \qquad \forall y \in B(x, \delta).$$

Elements in $\hat{N}_S(x)$ are called Fréchet normals to S at x.

Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be an extended real-valued function. The effective domain of f is defined by $\operatorname{dom}(f) := \{x \in X : f(x) < +\infty\}$ and the epigraph of f is defined by $\operatorname{epi}(f) := \{(x,\alpha) \in X \times \mathbb{R} : x \in \operatorname{dom}(f), \alpha \geq f(x)\}$. Let $x \in \operatorname{dom}(f)$. The Fréchet subdifferential of f at x is the set

$$\hat{\partial} f(x) := \left\{\zeta \in X^* : \liminf_{y \to x} \frac{f(y) - f(x) - \langle \zeta, y - x\rangle}{\| y - x \|} \geq 0\right\}.$$

Equivalently, $\zeta \in \hat{\partial} f(x)$ if and only if for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\langle \zeta, y - x\rangle \leq f(y) - f(x) + \varepsilon \| y - x \|, \qquad \forall y \in B(x, \delta).$$

We call elements in $\hat{\partial} f(x)$ the Fréchet subgradients of f at x. The Fréchet subdifferential of f at x can also be defined as

$$\hat{\partial} f(x)=\left\{\zeta\in X^*:(\zeta,-1)\in\hat{N}_{\mathrm{epi}(f)}(x,f(x))\right\}.$$

The Fréchet singular subdifferential of f at x is the set $\hat{\partial}^\infty f(x)$ which is defined by

$$\hat{\partial}^\infty f(x)=\left\{\zeta\in X^*:(\zeta,0)\in\hat{N}_{\mathrm{epi}(f)}(x,f(x))\right\}.$$

In other words, $\zeta\in\hat{\partial}^\infty f(x)$ if and only if for any $\varepsilon>0$, there exists $\delta>0$ such that

$$\langle\zeta,y-x\rangle\le\varepsilon(\|y-x\|+|\beta-f(x)|),\qquad\forall y\in B(x,\delta),(y,\beta)\in\mathrm{epi}(f).$$

We call elements in $\hat{\partial}^\infty f(x)$ the Fréchet singular subgradients of f at x.

The support function $\rho_A:X^*\to(-\infty,\infty]$ of a subset A of X is defined as: for $\zeta\in X^*$

$$\rho_A(\zeta)=\sup_{x\in A}\langle\zeta,x\rangle.$$

2. Fréchet singular subdifferentials of the minimal time function

For simplicity of the presentation, we consider the minimal time function associated with a collection of two subsets of X. Throughout this section, $\mathcal{U}=\{U_1,U_2\}$ is a collection of two nonempty, bounded subsets U_1,U_2 of X. We always assume that $U_1\cap U_2\subset\{0\}$ and $U_1\cup U_2\neq\{0\}$, Ω is closed. We denote $M=\sup\{\|u\|:u\in\mathbb{U}\}$ where $\mathbb{U}=U_1\cup U_2$. The function $T_{\mathcal{U},\Omega}$ is now written as:

$$T_{\mathcal{U},\Omega}(x)=\inf\left\{t_1+t_2:t_1,t_2\ge 0\,\text{and}\left(x+t_1U_1+t_2U_2\right)\cap\Omega\neq\varnothing\right\}.\qquad(2.1)$$

For $t\ge 0$, we define

$$\mathcal{R}(t):=\{x\in x:T_{\mathcal{U},\Omega}(x)\le t\},$$

and

$$\mathcal{R}:=\{x\in X:T_{\mathcal{U},\Omega}(x)<\infty\}.$$

Our first result is stated as follows.

Theorem 2.1. *Let* $x_0\in\Omega$. *We have*

$$\hat{\partial}^\infty T_{\mathcal{U},\Omega}(x_0)=\hat{N}_\Omega(x_0)\cap\{\zeta\in X^*:\max\{\rho_{U_1}(-\zeta),\rho_{U_2}(-\zeta)\}\le 0\}.\qquad(2.2)$$

Proof. Let $\zeta\in\hat{\partial}^\infty T_{\mathcal{U},\Omega}(x_0)$. Then, for any $\varepsilon>0$, there exists $\delta>0$ such that

$$\langle\zeta,y-x_0\rangle\le\varepsilon(\|y-x_0\|+\beta),\quad\forall y\in B(x_0,\delta),(y,\beta)\in\mathrm{epi}(T_{\mathcal{U},\Omega}).\qquad(2.3)$$

It follows that

$$\langle\zeta,y-x_0\rangle\le\varepsilon\|y-x_0\|,\quad\forall y\in\Omega\cap B(x_0,\delta).$$

This means that $\zeta \in \hat{N}_\Omega(x_0)$.

Let $u \in \mathbb{U}$ be arbitrary and let $\lambda > 0$ be sufficiently small such that $y := x_0 - \lambda u \in B(x_0, \delta)$. Then, we have $T_{\mathcal{U},\Omega}(y) \le \lambda$. From (2.3), one has

$$\langle \zeta, -\lambda u \rangle \le \varepsilon(\| -\lambda u \| + \lambda)$$

Dividing both sides of the latter inequality by $\lambda > 0$, we get

$$\langle \zeta, -u \rangle \le \varepsilon(\| u \| + 1).$$

Letting $\varepsilon \to 0^+$, we have $\langle \zeta, -u \rangle \le 0$. Since $u \in \mathbb{U}$ is arbitrary, $\rho_{\mathbb{U}}(-\zeta) \le 0$, or

$$\max\{\rho_{U_1}(-\zeta), \rho_{U_2}(-\zeta)\} \le 0.$$

Now, let $\zeta \in \hat{N}_\Omega(x_0)$ be such that

$$\max\{\rho_{U_1}(-\zeta), \rho_{U_2}(-\zeta)\} \le 0.$$

We shall prove that $\zeta \in \hat{\partial}^\infty T_{\mathcal{U},\Omega}(x_0)$. Assume to the contrary that $\zeta \notin \hat{\partial}^\infty T_{\mathcal{U},\Omega}(x_0)$ Then, there exist $C > 0$ and sequences $\{y_i\} \subset X$, $\{\beta_i\} \subset \mathbb{R}$ such that $y_i \to x_0$ as $i \to \infty$, and $\beta_i \ge T_{\mathcal{U},\Omega}(y_i), y_i \ne x_0$ and

$$\langle \zeta, y_i - x_0 \rangle \ge C(\| y_i - x_0 \| + \beta_i). \tag{2.4}$$

for all i. We have from (2.4) that

$$\langle \zeta, y_i - x_0 \rangle \ge C(\| y_i - x_0 \| + T_{\mathcal{U},\Omega}(y_i)), \qquad \forall i. \tag{2.5}$$

This yields

$$t_i := T_{\mathcal{U},\Omega}(y_i) \le \frac{1}{C} \| \zeta \| \| y_i - x_0 \|, \quad \forall i$$

and thus $t_i \to 0$ as $i \to \infty$.

Let $\eta > 0$. By the definition of $T_{\mathcal{U},\Omega}$, for each i, there exist $t_1^i, t_2^i \ge 0$, $w^i \in \Omega$, $u_1^i \in U_1$, $u_2^i \in U_2$ such that

$$t_i < t_1^i + t_2^i < t_i + \eta, \quad w^i = y_i + t_1^i u_1^i + t_2^i u_2^i$$

One has for all i that

$$\| w^i - x_0 \| = \| y_i + t_1^i u_1^i + t_2^i u_2^i - x_0 \| \le \| y_i - x_0 \| + (t_i + \eta) M.$$

Since $\eta > 0$ is arbitrary, $w^i \to x_0$ as $i \to \infty$. Let $\varepsilon > 0$. Since $\zeta \in \hat{N}_\Omega(x_0)$, for i large enough, we have

$$\langle \zeta, w^i - x_0 \rangle \le \varepsilon \| w^i - x_0 \|, \tag{2.6}$$

For i large enough,

$$\begin{aligned} C(\| y_i - x_0 \| + t_i) &\le \langle \zeta, y_i - x_0 \rangle = \langle \zeta, w^i - t_1^i u_1^i - t_2^i u_2^i - x_0 \rangle \\ &= \langle \zeta, w^i - x_0 \rangle + t_1^i \langle -\zeta, u_1^i \rangle + t_2^i \langle -\zeta, u_2^i \rangle \\ &\le \varepsilon \| w^i - x_0 \| \quad (\text{as } \rho_{U_i}(-\zeta) \le 0). \end{aligned}$$

Hence,

$$C \le \varepsilon \frac{\| w^i - x_0 \|}{\| y_i - x_0 \| + t_i}, \tag{2.7}$$

for i large enough. We claim that the sequence $\left\{ \frac{\| w^i - x_0 \|}{\| y_i - x_0 \| + t_i} \right\}$ is bounded. Assume to the contrary that the sequence is not bounded. Then, without loss of generality, there exists i_0 such that for $i > i_0$, we have

$$\frac{\| w^i - x_0 \|}{\| y_i - x_0 \| + t_i} > M + 2. \tag{2.8}$$

That is, for $i > i_0$,

$$(M+2)(\| y_i - x_0 \| + t_i) \le \| w^i - x_0 \| \le \| y_i - x_0 \| + (t_i + \varepsilon)M$$

Let $\varepsilon \to 0^+$, one has

$$(M+2)(\| y_i - x_0 \| + t_i) \le \| y_i - x_0 \| + t_i M$$

for all i sufficiently large. This implies that

$$(M+1) \| y_i - x_0 \| \le -2t_i < 0$$

for all $i > i_0$ large enough. This is a contradiction.

Set

$$Q = \sup_i \left\{ \frac{\| w^i - x_0 \|}{\| y_i - x_0 \| + t_i} \right\}.$$

From (2.7), we have $C \le \varepsilon Q$. Let $\varepsilon \to 0^+$, one gets $C \le 0$ which is a contradiction. Thus, $\zeta \in \hat{\partial}^\infty T_{\mathcal{U},\Omega}(x_0)$. This ends the proof.

Next, we give the formula for computing Fréchet singular subdifferentials of the minimal time function at a point outside the target. For that aim, we need the following result.

Proposition 2.1. *Let $x_0 \in X$ be such that $0 < r := T_{\mathcal{U},\Omega}(x_0) < +\infty$. If $\zeta \in \hat{N}_{\mathcal{R}(r)}(x_0)$ then we have $\rho_{\mathbb{U}}(-\zeta) \ge 0$.*

Proof. Since $\zeta \in \hat{N}_{\mathcal{R}(r)}(x_0)$, for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\langle \zeta, y - x_0 \rangle \le \varepsilon \| y - x_0 \|. \tag{2.9}$$

for all $y \in B(x_0, \delta) \cap \mathcal{R}(r)$.

Since $r = T_{\mathcal{U},\Omega}(x_0) < +\infty$, by the definition of $T_{\mathcal{U},\Omega}$, for $0 < \gamma < r/2$, there exist $t_1, t_2 \ge 0$, $w \in \Omega$, $u_1 \in U_1$, $u_2 \in U_2$ such that $r < t_1 + t_2 < r + \gamma$ and $w = x_0 + t_1 u_1 + t_2 u_2$. Without loss of generality, we may assume that $t_1 \ge t_2$ and $t_1 > 0$. Then, $t_1 > r/2$.

We take $r/2 < \eta < \max\{r, \delta/M\}$ and let $z = x_0 + \eta u_1$. It is easy to see that $z \in B(x_0, \delta)$. Moreover, since

$$z + (t_1 - \eta)u_1 + t_2 u_2 = w \in \Omega,$$

we have

$$T_{\mathcal{U},\Omega}(z) \le t_1 - \eta + t_2 < r + \gamma - \eta \le r.$$

It means that $z \in B(x_0, \delta) \cap \mathcal{R}(r)$. From (2.9), one has $\langle \zeta, \eta u_1 \rangle \le \varepsilon \| \eta u_1 \|$. Equivalently,

$$\langle \zeta, u_1 \rangle \le \varepsilon \| u_1 \|.$$

Let $\varepsilon \to 0+$, we get $\langle \zeta, u_1 \rangle \le 0$. Therefore, $\rho_U(-\zeta) \ge 0$. This ends the proof.

Theorem 2.2. *Let* U_1 *and* U_2 *be convex and* $x_0 \in X$ *such that* $0 < r := T_{\mathcal{U},\Omega}(x_0) < \infty$. *Then,*

$$\hat{\partial}^\infty T_{\mathcal{U},\Omega}(x_0) = \hat{N}_{\mathcal{R}(r)}(x_0) \cap \{\zeta \in X^* : \max\{\rho_{U_1}(-\zeta), \rho_{U_2}(-\zeta)\} = 0\}. \tag{2.10}$$

Proof. Assume that $\zeta \in \hat{\partial}^\infty T_{\mathcal{U},\Omega}$. Then, for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\langle \zeta, y - x_0 \rangle \le \varepsilon(\| y - x_0 \| + | \beta - r |), \quad \forall y \in B(x_0, \delta), (y, \beta) \in \mathrm{epi}(T_{\mathcal{U},\Omega}). \tag{2.11}$$

It follows that $\langle \zeta, y - x_0 \rangle \le \varepsilon \| y - x_0 \|, \forall y \in \mathcal{R}(r) \cap B(x_0, \delta),$ that is, $\zeta \in \hat{N}_{\mathcal{R}(r)}(x_0)$.

Since $r = T_{\mathcal{U},\Omega}(x_0) < +\infty$, for $0 < \gamma < r^2/4$, there exist $t_1, t_2 \ge 0$, $w \in \Omega$, $u_1 \in U_1$, $u_2 \in U_2$ such that $r < t_1 + t_2 < r + \gamma$, $w = x_0 + t_1 u_1 + t_2 u_2$.

Let $u \in \mathbb{U}$ be arbitrary and $\lambda > 0$. Assume that $u \in U_1$. Then, by the convexity of U_1, one has $w \in x_0 - \lambda u + \lambda U_1 + t_1 U_1 + t_2 U_2 = x_0 - \lambda u + (t_1 + \lambda) U_1 + t_2 U_2$.

Thus, $T_{\mathcal{U},\Omega}(x_0 - \lambda u) \le t_1 + t_2 + \lambda < r + \gamma + \lambda$. For λ sufficiently small, we have $x_0 - \lambda u \in B(x_0, \delta)$. By (2.11), one has $\langle \zeta, -\lambda u \rangle \le \varepsilon(\| -\lambda u \| + | \lambda + \gamma |)$.

Let $\gamma \to 0^+$, we get $\lambda \langle -\zeta, u \rangle \le \lambda \varepsilon(\| u \| + 1)$.

Divide both sides of the latter inequality by $\lambda > 0$ and then let $\varepsilon \to 0^+$, we obtain $\langle -\zeta, u \rangle \le 0$. Similarly, if $u \in U_2$, we can also show that $\langle -\zeta, u \rangle \le 0$. Since $u \in \mathcal{U}$ is arbitrary, $\rho_{\mathbb{U}}(-\zeta) \le 0$. Combining with Proposition 2.1, we have $\rho_{\mathbb{U}}(-\zeta) = 0$.

Conversely, let $\zeta \in \hat{N}_{\mathcal{R}(r)}(x_0)$ be such that $\rho_{\mathbb{U}}(-\zeta) = 0$. We show that $\zeta \in \hat{\partial}^\infty T_{\mathcal{U},\Omega}(x_0)$. Assume to the contrary that $\zeta \notin \hat{\partial}^\infty T_{\mathcal{U},\Omega}(x_0)$. Then, there exists $\alpha > 0$ and sequences $\{y_i\} \subset X$, $\{\beta_i\} \subset \mathbb{R}$ such that $y_i \to x_0$, $y_i \ne x_0$, $r_i := T_{\mathcal{U},\Omega}(y_i) \le \beta_i$ and\

$$\langle \zeta, y_i - x_0 \rangle \ge \alpha(\| y_i - x_0 \| + | \beta_i - r |), \quad \forall i. \tag{2.12}$$

We consider two cases:

Case 1. There exists a subsequence of $\{y_i\}$ which we still denote by $\{y_i\}$ such that $T_{\mathcal{U},\Omega}(y_i) \le T_{\mathcal{U},\Omega}(x_0)$ for all i. In this case, $y_i \in \mathcal{R}(r)$ for all i. Since $\zeta \in \hat{N}_{\mathcal{R}(r)}(x_0)$, for any $\varepsilon > 0$,

$$\langle \zeta, y_i - x_0 \rangle \le \varepsilon \| y_i - x_0 \|$$

for i large enough. Combining with (2.12), one gets

$$\varepsilon \| y_i - x_0 \| \ge \alpha(\| y_i - x_0 \| + | \beta_i - r |) \ge \alpha \| y_i - x_0 \|$$

which implies $\alpha \le \varepsilon$. This is a contradiction.

Case 2. There exists a subsequence of $\{y_i\}$ which is still denoted by $\{y_i\}$ such that $T_{\mathcal{U},\Omega}(y_i) > T_{\mathcal{U},\Omega}(x_0)$ for all i. By (2.12), $r_i = T_{\mathcal{U},\Omega}(y_i) < +\infty$ and

$$0 < r_i - r \le \beta_i - r \le \frac{1}{\alpha} \| \zeta \| \| y_i - x_0 \|$$

for all i. It implies that $r_i \to r$ as $i \to \infty$. Hence, we may assume that $5(r_i - r) < r$ for all i. By the definition of the minimal time function, for each i, there exist $t_1^i, t_2^i \ge 0$, $w_i \in \Omega$, $u_1^i \in U_1$, $u_2^i \in U_2$ such that

$$r_i < t^i - 1 + t_2^i < 2r_i - r, \quad w_i = y_i + t_1^i u_1^i + t_2^i u_2^i.$$

Without loss of generality, we may assume that $t_1^i \ge t_2^i$. Then, for all i,

$$t_1^i \ge \frac{1}{2}(t_1^i + t_2^i) > \frac{1}{2} r_i > 3(r_i - r).$$

For each i, let $\gamma_i \in (2r_i - 2r, 3r_i - 3r)$. Then,

$$w_i = y_i + \gamma_i u_i^i + (t_1^i - \gamma_i) u_1^i + t_2^i u_2^i.$$

Thus, $T_{\mathcal{U},\Omega}(y_i + \gamma_i u_1^i) \le t_1^i - \gamma_i + t_2^i < 2r_i - r - (2r_i - 2r) = r.$

This means that $y_i + \gamma_i u_1^i \in \mathcal{R}(r)$. Moreover,

$$\| y_i + \gamma_1^i u_i - x_0 \| \le \| y_i - x_0 \| + 3(r_i - r)M \to 0 \quad \text{as} i \to \infty.$$

That is, for i large enough, $y_i + \gamma_1^i u_i \in \mathcal{R}(r) \cap B(x_0, \delta)$. Let $\varepsilon > 0$. Since $\zeta \in \hat{N}_{\mathcal{R}(r)}(x_0)$, for i sufficiently large,

$$\langle \zeta, y_i + \gamma_1^i u_i - x_0 \rangle \le \varepsilon \| y_i + \gamma_1^i u_i - x_0 \|.$$

Since $\rho_{\mathbb{U}}(-\zeta) = 0$,

$$\langle \zeta, y_i - x_0 \rangle \le \varepsilon \| y_i + \gamma_i u_1^i - x_0 \| + \gamma_i \langle -\zeta, u_1^i \rangle \le \varepsilon \| y_i + \gamma_i u_1^i - x_0 \|.$$

Combining with (2.12), one has

$$\alpha(\| y_i - x_0 \| + | r_i - r |) \le \alpha(\| y_i - x_0 \| + | \beta_i - r |) \le \varepsilon \| y_i + \gamma_i u_1^i - x_0 \|.$$

Thus,

$$\begin{aligned}\alpha &\le \varepsilon \frac{\| y_i + \gamma_i u_1^i - x_0 \|}{\| y_i - x_0 \| + | r_i - r |} \\ &\le \varepsilon \frac{\| y_i - x_0 \| + 3M | r_i - r |}{\| y_i - x_0 \| + | r_i - r |} \\ &\le (3M + 1)\varepsilon.\end{aligned}$$

Letting $\varepsilon \to 0^+$, we have $\alpha \le 0$. This is a contradiction. Therefore, $\zeta \in \hat{\partial}^\infty T_{\mathcal{U},\Omega}(x_0)$. The proof is complete.

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