FRÉCHET SINGULAR SUBDIFFERENTIALS OF THE MINIMAL TIME FUNCTION ASSOCIATED WITH A COLLECTION OF SETS

Nguyen Van Luong¹, Nguyen Thi Xuan¹, Nguyen Thi Nga², Van Thi Trang³

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Abstract: In this paper, we present the formulas for computing the Fréchet singular subdifferentials of the minimal time function associated with a collection of subsets in normed spaces.

Keywords: Minimal time function, Fréchet singular subdifferentials, Normed spaces.

1. Introduction and preliminaries

Let X be a normed space, F and Ω be two nonempty subsets of X. The minimal time function with the constant dynamics F and the target set Ω is defined by

$$
T_0^F(x) := \inf\{t \ge 0 : (x + tF) \cap \Omega \ne \emptyset\}, \quad x \in X. \tag{1.1}
$$

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 is pap The minimal time function T_{Ω}^F plays an important role in variational analysis since its covers three crucial functions in variational analysis: the distance function, the Minkowski function and the indicator function. Variational analysis and subdifferentials of the minimal time function with a convex dynamics containing the origin in its interior, in Hilbert spaces were first investigated by Colombo and Wolenski in [5] [6]. Later, the function has been studied extensively by many researchers; see, e.g., [2] [3] [4] [7] [8] [9] [12] [13] [16] [19] [21]. He [8] studied subdifferentials of the minimal time function in Banach spaces. These results were extended to the setting of normed spaces by Jiang and He in [9] and then improved by Mordukhovich and Nam in [12] [13]. Bounkhel investigated subdifferential calculus of the minimal time function in Hausdorff topological vector spaces in [2] [3]. Applications of variational analysis and generalized differentiations of the minimal time function to generalized location problems were presented in [13] [14] [15] [16] [17] [18] [20] and references therein. The minimal time function T_0^F plays an important role in variational analysis
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The notion of the minimal time function associated with a collection of set was recently introduced in [11]. This new function contains the classical minimal time function as a special case. More precisely, let m be a positive integer and let $\{1, \dots, U_m\}$ be a collection of m nonempty subsets U_1, \dots, U_m of X and Ω a nonempty subset of X. The minimal time function associated with the collection \mathcal{U} to

¹ Faculty of Natural Sciences, Hong Duc University; Email: nguyenvanluong@hdu.edu.vn

² Faculty of Primary Education, Hong Duc University

³ Ouang Xuong 4 High School, Quang Xuong District, Thanh Hoa Province

$$
T_{\mathcal{U},\Omega}(x) := \inf \left\{ t_1 + \dots + t_m : t_1, \dots, t_m \ge 0 \text{ and } \left(x + t_1 U_1 + \dots + t_m U_m \right) \cap \Omega \neq \varnothing \right\}
$$

Fréchet singular subdifferentials of the minimal time function associated with a collection of sets
 $T_{U,\Omega}(x) := \inf \{t_1 + \cdots + t_m : t_1, \cdots, t_m \ge 0 \text{ and } (x + t_1U_1 + \cdots + t_mU_m) \cap \Omega \neq \emptyset\}.$

it is obvious that if $U_1 = F$ and $U_2 = \cdots$ réchet singular subdifferentials of the minimal time function associated with a collection of sets
 $(x) := \inf \{t_1 + \cdots + t_m : t_1, \cdots, t_m \ge 0 \text{ and } (x + t_1 U_1 + \cdots + t_m U_m) \cap \Omega \neq \emptyset\}.$

bbvious that if $U_1 = F$ and $U_2 = \cdots = U_m = \{0\}$, t Fréchet singular subdifferentials of the minimal time function associated with a collection of sets
 $T_{U,\Omega}(x) := \inf \{t_1 + \cdots + t_m : t_1, \cdots, t_m \ge 0 \text{ and } (x + t_1U_1 + \cdots + t_mU_m) \cap \Omega \neq \emptyset\}.$

It is obvious that if $U_1 = F$ and $U_2 = \cdots$ f the minimal time function associated with a collection of sets
 $t, t_m \ge 0$ and $(x + t_1U_1 + \cdots + t_mU_m) \cap \Omega \ne \emptyset$.
 $U_2 = \cdots = U_m = \{0\}$, then $T_{U,\Omega}$ becomes the usual

1.1). Let $x \in X$. From the definition of the minimal
 , then $T_{\mathcal{U},\Omega}$ becomes the usual minimal time function T_{Ω}^{F} defined in (1.1). Let $x \in X$. From the definition of differentials of the minimal time function associated with a collection of sets
 $\cdots + t_m : t_1, \cdots, t_m \ge 0$ and $(x + t_1U_1 + \cdots + t_mU_m) \cap \Omega \neq \emptyset$.
 $U_1 = F$ and $U_2 = \cdots = U_m = \{0\}$, then $T_{U,\Omega}$ becomes the usual

defined in (1. time function $T_{\mathcal{U}, \Omega}$, we see that if $T_{\mathcal{U}, \Omega}(x) < \infty$, then it is the smallest t tials of the minimal time function associated with a collection of sets
 $t_1, \dots, t_m \ge 0$ and $\left(x + t_1U_1 + \dots + t_mU_m\right) \cap \Omega \neq \emptyset$.

and $U_2 = \dots = U_m = \{0\}$, then $T_{U,\Omega}$ becomes the usual

d in (1.1). Let $x \in X$. From the def the minimal time function associated with a collection of sets
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 $= \cdots = U_m = {0}$, then $T_{U,\Omega}$ becomes the usual
 ≥ 0 . Let $x \in X$. From the definition of the minimal
 $< \infty$, target Ω using at most one direction in each set U_1, \dots, U_m . It means that x may be steered aal time function associated with a collection of sets

nd $(x+t_1U_1 + \cdots + t_mU_m) \cap \Omega \neq \emptyset$.
 $= U_m = \{0\}$, then $T_{U,\Omega}$ becomes the usual
 $x \in X$. From the definition of the minimal

then it is the smallest time to steer x to the target Ω in a "zigzag" path. This contrasts with the case of the classical minimal time function as points are steered to the target along a straight path. It turns out that the new type of minimal time function is more flexible than the classical one and it can be used to model problems that the classical one cannot. By careful adaptation of existing results for the classical minimal time function, in [11], we present various basic properties of the new minimal time function. These properties (which include, among others, lower semicontinuity, Lipschitz continuity, convexity, principle of optimality and subdifferential calculus) were then utilized to study a location problem. The aim of this paper is to continue investigating the minimal time function associated with a collection of sets by proving the formulas for computing Fréchet singular subdifferentials of the function. it is obvious that if $\sigma_1 = r^2$ and $\sigma_2 = \cdots = \sigma_m = \nu_f$, then $i_{kL,\Omega}$ becomes the usual
minimal time function T_L^S defined in (1.1). Let $x \in X$. From the definition of the minimal
time function T_L^S defined in (1. minimal time function T_{α}^{P} defined in (1.1). Let $x \in X$. From the definition of the minimal
time function $T_{\alpha,0}^{P}$, we see that if $T_{\alpha,0}(x) < \infty$, then it is the smallest time to steer x to the
target Ω usin time tunction $I_{U,D}$, we see that if $I_{U,D}(x) \ll \infty$, then it is the smallest time to steer *x* to the target Ω using at most one direction in each set U_1, \cdots, U_m . It means that *x* may be steered to the target Ω in a o the target Ω in a "zigzag" path. This contrasts with the case of the cl
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We now recall some basic concepts of nonsmooth analysis [1] [10]. Let X be utilized to study a location problem. The aim of this paper is to continue
inimal time function associated with a collection of sets by proving the
ting Fréchet singular subdifferentials of the function.
all some basic co are called Fréchet normals to S at x.

Let $f: X \rightarrow \mathbb{R} \cup \{+\infty\}$ be an extended real-valued function. The set of the function of sets by proving the

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 formulas for computing Fréchet singular subdifferentials of the function.

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normed space and X' be the topological dual space of X. We denote by We now recall some basic concepts of nonsmooth analysis [1] [10]. Let X be

normed space and X' be the topological dual space of X. We denote by $||\cdot||$ the norm

in X and by $\langle \cdot, \cdot \rangle$ the dual par between X and X'. We a

, is the set $\hat{M}(x) = \int z - Y^* \cdot \limsup \langle \zeta, y - x \rangle > 0$

$$
\hat{N}_S(x) := \left\{ \zeta \in X^* : \limsup_{S \ni y \to x} \frac{\langle \zeta, y - x \rangle}{\|y - x\|} \le 0 \right\}.
$$

In other words, $\zeta \in \hat{N}_s(x)$ if and only if for any $\varepsilon > 0$, there exists $\delta > 0$ such that

Elements in $\hat{N}_s(x)$ are called Fréchet normals to S at x.

subdifferential of f at x is the set The dual pair between X and X. We also denote by $||\cdot||$ the dual norm
 $y B(x, r)$ the open ball of radius $r > 0$ centered at x and $\mathbb{B} = B(0,1)$.

the a closed set and let $x \in S$. The Fréchet normal cone to S at x, writte $B(x, r)$ the open ball of radius $r > 0$ centered at x and $\mathbb{B} = B(0,1)$.

e a closed set and let $x \in S$. The Fréchet normal cone to S at x , written
 $\hat{N}_S(x) := \left\{ \zeta \in X^* : \limsup_{S \ni r \to x} \frac{(\zeta, y - x)}{\|y - x\|} \le 0 \right\}.$
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 $:X^*$: $\limsup_{S \ni \to x} \frac{\langle \zeta, y - x \rangle}{\|y - x\|} \le 0$.

Ad only if for any $\varepsilon > 0$, there exists $\delta > 0$ such the dual part between X and X . We also denote by $||\cdot||$ the dual norm $B(x, r)$ the open ball of radius $r > 0$ centered at x and $\mathbb{B} = B(0,1)$.

be a closed set and let $x \in S$. The Fréchet normal cone to S at x, written
 • 0 centered at *x* and $B = B(0,1)$.

réchet normal cone to *S* at *x*, written
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alued function. The effective do and by $\langle \cdot, \cdot \rangle$ the dual pair between X and X'. We also denote by $|| \cdot ||$ the dual norm

Denote by $B(x, r)$ the open ball of radius $r > 0$ centered at x and $\mathbb{B} = B(0,1)$.

Let $S \subset X$ be a closed set and let $x \in S$. T $\hat{N}_s(x) := \left\{ \zeta \in X^* : \limsup_{S \ni y \to x} \frac{\langle \zeta, y - x \rangle}{\|y - x\|} \le 0 \right\}.$

ords, $\zeta \in \hat{N}_s(x)$ if and only if for any $s > 0$, there exists $\delta > 0$ such that
 $\langle \zeta, y - x \rangle \le \varepsilon \|y - x\|, \quad \forall y \in B(x, \delta).$
 $\hat{N}_s(x)$ are called Fréc

$$
\hat{\partial}f(x) := \left\{ \zeta \in X^* : \liminf_{y \to x} \frac{f(y) - f(x) - \langle \zeta, y - x \rangle}{\|y - x\|} \ge 0 \right\}.
$$

Hong Duc University Journal of Science, E7, vol.12, p.(57-65), 2022
We call elements in $\hat{\partial}f(x)$ the Fréchet subgradients of f at x. The Fréchet
ferential of f at x can also be defined as
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is the set $\hat{\partial}^{\infty} f(x)$ which is defined by
 $\begin{aligned}\n\begin{cases}\n\text{Equation (x, f(x))}\n\end{cases}.\n\end{aligned}$ Hong Duc University Journal of Science, E7, vol.12, p.(57-65), 2022

tts in $\hat{\partial}f(x)$ the Fréchet subgradients of f at x. The Fréchet

x can also be defined as
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\mathcal{F}(x) \text{ which is defined by} \\
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\hat{\partial}f(x) = \left\{ \zeta \in X^* : (\zeta, -1) \in \hat{N}_{\text{epi}(f)}(x, f(x)) \right\}.
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The Fréchet singular subdifferential of f at x is the set $\hat{\partial}^{\infty} f(x)$ which is defined by

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\hat{\partial}^{\infty} f(x) = \left\{ \zeta \in X^* : (\zeta, 0) \in \hat{N}_{\text{epi}(f)}(x, f(x)) \right\}.
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 $\exp(f(x, f(x)))$.

is the set $\hat{\partial}^{\infty} f(x)$ which is defined by
 $\exp(f(x, f(x)))$.

my $\varepsilon > 0$, there exists $\delta > 0$ such that
 $\forall y \in B(x, \delta), (y, \beta)$ Hong Duc University Journal of Science, E7, vol.12, p.(57-65), 2022

(a) $\hat{\partial}f(x)$ the Fréchet subgradients of f at x. The Fréchet

(a) and also be defined as
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this in** $\hat{\partial}f(x)$ **the Fréchet subgradients of f at x. The Fréchet
** $\hat{\partial}f(x) = \{ \zeta \in X^* : (\zeta, -1) \in \hat{N}_{\varphi(i,f)}(x, f(x)) \}.$ **

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We call elements in $\hat{\partial}f(x)$ the Fréchet subgradients of f at x. The Fréchet
Ferential of f at x can also be defined as
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e call elements in $\hat{\partial}f(x)$ the Fréchet subgradients of f at x. The Fréchet

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We call elements in $\hat{\partial}^{\infty} f(x)$ the Fréchet singular subgradients of f at x.

The support function $\rho_A: X^* \to (-\infty, \infty]$ of a subset A of X is defined as: for $\zeta \in X^*$

$$
\rho_{A}(\zeta) = \sup_{x \in A} \langle \zeta, x \rangle.
$$

2. Fréchet singular subdifferentials of the minimal time function

For simplicity of the presentation, we consider the minimal time function associated Using the Fréchet singular subdifferential of f at x is the set $\hat{\sigma}^* f(x)$ which is defined by
 $\hat{\sigma}^* f(x) = \{ \zeta \in X^* : (\zeta, 0) \in \hat{N}_{\text{sp}(f)}(x, f(x)) \}.$

In other words, $\zeta \in \hat{\sigma}^* f(x)$ if and only if for any $\varepsilon > 0$ The Fréchet singular subdifferential of f at x is the set $\partial^x f(x)$ which is defined by
 $\partial^x f(x) = \{\zeta \in X^* : (\zeta, 0) \in \hat{N}_{\varphi_0(\zeta)}, (x, f(x))\}.$

In other words, $\zeta \in \hat{\partial}^* f(x)$ if and only if for any $\varepsilon > 0$, there ex $\frac{\partial^{\alpha} f(x)}{\partial x} = \left\{ \zeta \in X' : (\zeta, 0) \in \hat{N}_{q\circ i,f}(x,f(x)) \right\}.$

In other words, $\zeta \in \hat{\mathcal{C}}^{\alpha} f(x)$ if and only if for any $\varepsilon > 0$, there exists $\delta > 0$ such that
 $\langle \zeta, y-x \rangle \leq \varepsilon (\|y-x\| + |\beta - f(x)|), \qquad \forall y \in B(x, \delta), (y, \beta) \in \text{$ In other words, $\zeta \in \hat{\partial}^{\infty} f(x)$ if and only if for any $\varepsilon > 0$, there exists $\delta > 0$ su
 $\langle \zeta, y-x \rangle \leq \varepsilon(||y-x||+|\beta-f(x)|), \quad \forall y \in B(x, \delta), (y, \beta) \in \text{epi}(f).$

We call elements in $\hat{\partial}^{\infty} f(x)$ the Fréchet singular subgradi $\mathbb{U} = U_1 \cup U_2$. The function $T_{\mathcal{U},\Omega}$ is now written as: $\langle \zeta, y - x \rangle \leq \varepsilon(||y - x|| + |\beta - f(x)|), \quad \forall y \in B(x, \delta), (y, \beta) \in \text{epi}(f).$
We call elements in $\hat{\sigma}^{\alpha} f(x)$ the Fréchet singular subgradients of f at x.
The support function $\rho_A: X^* \to (-\infty, \infty]$ of a subset A of X is defined as: fo $\zeta', y-x \le \varepsilon(||y-x|| + |\beta - f(x)|), \quad \forall y \in B(x, \delta), (y, \beta) \in \text{epi}(f).$
call elements in $\hat{\sigma}^c f(x)$ the Fréchet singular subgradients of f at x.
support function $\rho_4: X^* \to (-\infty, \infty]$ of a subset A of X is defined as: for
 $\rho_4(\zeta) = \sup_{$ $\mathcal{P}_A(\zeta) = \sup_{x \in A} \langle \zeta, x \rangle$.

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i.e., $\mathcal{P}_A(\zeta) = \sup_{x \in A} \langle \zeta, x \rangle$.

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i.e., $\mathcal{U} = \{U_1, U_2\}$ is a

ded subsets U_1, U_2 of X . We always assume tha $\rho_A(\zeta) = \sup_{x \in A} \langle \zeta, x \rangle$.

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 $U_2 \neq \{0\}$, Ω is simplicity of the presentation, we consider the minimal time function associated
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of two nonempty, bounded subsets U_1, U_2 of X. We always assum For simplicity of the presentation, we consider the minimal time function associated

a collection of two subsets of X. Throughout this section, $\mathcal{U} = \{U_1, U_2\}$ is a

cetion of two nonempty, bounded subsets U_1, U_2 *x* collection of two subsets of X. Throughout this section, $\mathcal{U} = \{U_1, U_2\}$ is a
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 $U_2 \subset \{0\}$ and $U_1 \cup U_2 \neq \{0\}$, Ω is closed. We of two subsets of X. Throughout this section, $\mathcal{U} = \{U_1, U_2\}$ is a

nonempty, bounded subsets U_1, U_2 of X. We always assume that

d $U_1 \cup U_2 \neq \{0\}$, Ω is closed. We denote $M = \sup\{\|u\|: u \in \mathbb{U}\}$ where

func

$$
T_{\mathcal{U},\Omega}(x) = \inf \left\{ t_1 + t_2 : t_1, t_2 \ge 0 \text{ and } \left(x + t_1 U_1 + t_2 U_2 \right) \cap \Omega \ne \emptyset \right\}.
$$
\n
$$
\text{For } t \ge 0 \text{, we define}
$$
\n
$$
(2.1)
$$

$$
\mathcal{R}(t) := \{x \in x : T_{\mathcal{U}, \Omega}(x) \le t\},\
$$

and

$$
\mathcal{R} \coloneqq \{ x \in X : T_{\mathcal{U}, \Omega}(x) < \infty \}.
$$

Our first result is stated as follows.

Theorem 2.1. Let $x_0 \in \Omega$. We have

\n (c) of two nonempty, bounded subsets
$$
U_1, U_2
$$
 of X . We always assume that $U_2 \subset \{0\}$ and $U_1 \cup U_2 \neq \{0\}$, Ω is closed. We denote $M = \sup\{\|u\|: u \in \mathbb{U}\}$ where $U_1 \cup U_2$. The function $T_{U,\Omega}$ is now written as:\n

\n\n $T_{U,\Omega}(x) = \inf\{t_1 + t_2 : t_1, t_2 \geq 0 \text{ and } (x + t_1U_1 + t_2U_2) \cap \Omega \neq \emptyset\}$.\n

\n\n (2.1) For $t \geq 0$, we define\n

\n\n $\mathcal{R}(t) := \{x \in x : T_{U,\Omega}(x) \leq t\}$,\n

\n\n and\n

\n\n $\mathcal{R} := \{x \in X : T_{U,\Omega}(x) < \infty\}$.\n

\n\n Our first result is stated as follows.\n

\n\n Theorem 2.1. Let $x_0 \in \Omega$. We have\n

\n\n $\partial^{\infty} T_{U,\Omega}(x_0) = \hat{N}_{\Omega}(x_0) \cap \{\zeta \in X^* : \max\{\rho_{U_1}(-\zeta), \rho_{U_2}(-\zeta)\} \leq 0\}$.\n

\n\n Proof. Let $\zeta \in \partial^{\infty} T_{U,\Omega}(x_0)$. Then, for any $\varepsilon > 0$, there exists $\delta > 0$ such that\n

\n\n $\langle \zeta, y - x_0 \rangle \leq \varepsilon (\|y - x_0\| + \beta)$,\n $\forall y \in B(x_0, \delta), (y, \beta) \in \operatorname{epi}(T_{U,\Omega})$.\n

\n\n (1.3) It follows that\n

\n\n $\langle \zeta, y - x_0 \rangle \leq \varepsilon \|y - x_$

It follows that

$$
\langle \zeta, y - x_0 \rangle \le \varepsilon ||y - x_0||, \quad \forall y \in \Omega \cap B(x_0, \delta).
$$

This means that $\zeta \in \hat{N}_{\Omega}(x_0)$.

r subdifferentials of the minimal time function associated with a collection of sets
 $\zeta \in \hat{N}_{\Omega}(x_0)$.

arbitrary and let $\lambda > 0$ be sufficiently small such that

1. Then, we have $T_{\mathcal{U},\Omega}(y) \leq \lambda$. From (2.3), one Let $u \in \mathbb{U}$ be arbitrary and let $\lambda > 0$ be sufficiently small such that Fréchet singular subdifferentials of the minimal time function associated with a collection of sets

This means that $\zeta \in \hat{N}_{\Omega}(x_0)$.

Let $u \in \mathbb{U}$ be arbitrary and let $\lambda > 0$ be sufficiently small such that
 $y := x$ is eminimal time function associated with a collection of sets

let $\lambda > 0$ be sufficiently small such that
 $T_{\mathcal{U},\Omega}(y) \leq \lambda$. From (2.3), one has
 $\leq \varepsilon(||-\lambda u|| + \lambda)$

equality by $\lambda > 0$, we get
 $\geq \varepsilon(||u|| + 1)$. imal time function associated with a collection of sets
 $\lambda > 0$ be sufficiently small such that
 $(y) \le \lambda$. From (2.3), one has
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ity by $\lambda > 0$, we get
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 K_0).

y and let $\lambda > 0$ be sufficiently small such that

we have $T_{\mathcal{U},\Omega}(y) \leq \lambda$. From (2.3), one has
 $\langle \zeta, -\lambda u \rangle \leq \varepsilon(||-\lambda u|| + \lambda)$

la ials of the minimal time function associated with a collection of sets

and let $\lambda > 0$ be sufficiently small such that

have $T_{\mathcal{U},\Omega}(y) \leq \lambda$. From (2.3), one has
 $\langle -\lambda u \rangle \leq \varepsilon(||-\lambda u|| + \lambda)$

tter inequality by $\lambda >$ *r* subdifferentials of the minimal time function associated with a collection of sets $\zeta \in \hat{N}_{\Omega}(x_0)$.

arbitrary and let $\lambda > 0$ be sufficiently small such that

1). Then, we have $T_{U,\Omega}(y) \leq \lambda$. From (2.3), one h

Dividing both sides of the latter inequality by $\lambda > 0$, we get

$$
\langle \zeta, -u \rangle \leq \varepsilon(||u||+1).
$$

Letting $\varepsilon \to 0^+$, we have $\langle \zeta, -u \rangle \le 0$. Since $u \in \mathbb{U}$ is arbitrary, $\rho_{\mathbb{U}}(-\zeta) \le 0$, or

$$
\max\{\rho_{U_1}(-\zeta),\rho_{U_2}(-\zeta)\}\leq 0
$$

Now, let $\zeta \in \hat{N}_{\Omega}(x_0)$ be such that

$$
\max \{ \rho_{U_1}(-\zeta), \rho_{U_2}(-\zeta) \} \le 0.
$$

t singular subdifferentials of the minimal time function associated with a collection of sets

ns that $\zeta \in \hat{N}_{\Omega}(x_0)$.

U be arbitrary and let $\lambda > 0$ be sufficiently small such that
 $S(x_0, \delta)$. Then, we have T_{U $\begin{aligned} &\hat{\mathbf{W}}_{\Omega}(x_0) \text{ if } \hat{\mathbf{W}}_{\Omega}(x_0) \text{ if } \hat{\mathbf{W}}_{\Omega}(x_0) \text{ if } \hat{\mathbf{W}}_{\Omega}(x_0) \text{ if } \hat{\mathbf{W}}_{\Omega}(y) \leq \lambda \text{ if } \hat{\mathbf{W}}_{\$ Freehet singular subdifferentials of the minimal time function associated with a collection of sets

This means that $\zeta \in \hat{N}_{\Omega}(x_0)$.

Let $u \in \mathbb{U}$ be arbitrary and let $\lambda > 0$ be sufficiently small such that
 v_0 ifferentials of the minimal time function associated with a collection of sets
 $\hat{N}_{\Omega}(x_0)$.

itrary and let $\lambda > 0$ be sufficiently small such that

en, we have $T_{\{I,\Omega\}}(y) \leq \lambda$. From (2.3), one has
 $\langle \zeta, -\lambda u \rangle \$ be entials of the minimal time function associated with a collection of sets
 $\Omega(x_0)$.
 $\Omega(x_0)$ any and let $\lambda > 0$ be sufficiently small such that
 $\langle \zeta, -\lambda u \rangle \le \varepsilon(||-\lambda u|| + \lambda)$

the latter inequality by $\lambda > 0$, we ge the function associated with a collection of sets
 $\lambda > 0$ be sufficiently small such that
 $\alpha(y) ≤ \lambda$. From (2.3), one has
 $\varepsilon(||-\lambda u||+\lambda)$

ality by $\lambda > 0$, we get
 $\varepsilon(||u||+1)$.
 \therefore Since $u ∈ \mathbb{U}$ is arbitrary, ρ a collection of sets

anall such that

s
 $y_j(-\zeta) \le 0$, or
 $\zeta \notin \hat{\partial}^{\infty} T_{\mathcal{U}, \Omega}(x_0)$

at $y_i \to x_0$ as

(2.4) We shall prove that $\zeta \in \hat{\partial}^{\infty} T_{\mathcal{U},\Omega}(x_0)$. Assume to the contrary that $\zeta \notin \hat{\partial}^{\infty} T_{\mathcal{U},\Omega}(x_0)$ This means that $\zeta \in \hat{N}_\Omega(x_0)$.

Let $u \in \mathbb{U}$ be arbitrary and let $\lambda > 0$ be sufficiently small such that $y := x_0 - \lambda u \in B(x_0, \delta)$. Then, we have $T_{\lambda(\Omega)}(y) \leq \lambda$. From (2.3), one has $\langle \zeta, -\lambda u \rangle \leq \alpha (\| -\lambda u \| + \lambda)$
 such that
 $(y) \le 0$, or
 $\hat{\sigma}^{\infty} T_{\mathcal{U}, \Omega}(x_0)$
 $y_i \to x_0$ as

(2.4) This means that $\zeta \in \hat{N}_{\Omega}(x_0)$.

Let $u \in \mathbb{U}$ be arbitrary and let $\lambda > 0$ be sufficiently small such that
 $y := x_0 - \lambda u \in B(x_0, \delta)$. Then, we have $T_{\ell(\Omega)}(y) \leq \lambda$. From (2.3), one has
 $\langle \zeta, -\lambda u \rangle \leq \alpha (\| -\lambda u \| + \lambda$ $i \rightarrow \infty$, and $\beta_i \geq T_{\mathcal{U},\Omega}(y_i), y_i \neq x_0$ and means that $\zeta \in N_{\Omega}(x_0)$.
 $u \in U$ be arbitrary and let $\lambda > 0$ be sufficiently small such that
 $u \in B(x_0, \delta)$. Then, we have $T_{i\ell,\Omega}(y) \leq \lambda$. From (2.3), one has
 $\langle \zeta, -\lambda u \rangle \leq \varepsilon (\| -\lambda u \| + \lambda)$

ling both sides of Dividing both sides of the latter inequality by $\lambda > 0$, we get
 $\langle \zeta, -\lambda u \rangle \le \varepsilon(||-\lambda u|| + \lambda)$

Dividing both sides of the latter inequality by $\lambda > 0$, we get
 $\langle \zeta, -u \rangle \le 0$. Since $u \in U$ is arbitrary, $\rho_U(-\zeta) \le 0$, $\exists z_0 = At \in B(x_0, \sigma)$. Inen, we have $t_{U,\Omega}(y) \leq \lambda$. From (2.5), one has
 $\langle \zeta, -\lambda u \rangle \leq \varepsilon(|-\lambda u|) + \lambda \rangle$

Dividing both sides of the latter inequality by $\lambda > 0$, we get
 $\langle \zeta, -u \rangle \leq \varepsilon (||u|| + 1)$.

Letting $\varepsilon \to 0^$ $\langle \zeta, -u \rangle \leq \varepsilon(\|u\| + 1).$
 $\langle \zeta, -u \rangle \leq 0$. Since $u \in \mathbb{U}$ is arbitrary, $\rho_{\mathbb{U}}(-\zeta) \leq 0$, or
 $\arg \{\rho_{U_i}(-\zeta), \rho_{U_2}(-\zeta)\} \leq 0.$

uch that
 $\arg \{\rho_{U_i}(-\zeta), \rho_{U_2}(-\zeta)\} \leq 0.$

sequences $\{y_i\} \subset X$, $\{\beta_i\$ $\langle \zeta, -u \rangle \le \varepsilon (\|u\|+1).$

chave $\langle \zeta, -u \rangle \le 0$. Since $u \in \mathbb{U}$ is arbitrary, $\rho_U(-\zeta) \le 0$, or
 $\max \{\rho_{U_1}(-\zeta), \rho_{U_2}(-\zeta)\} \le 0.$

(b) be such that
 $\max \{\rho_{U_1}(-\zeta), \rho_{U_2}(-\zeta)\} \le 0.$
 $\zeta \in \hat{\sigma}^{\infty} T_{U,\Omega}(x_0$ $\langle \zeta, -u \rangle \leq \varepsilon(||u|| + 1).$

ave $\langle \zeta, -u \rangle \leq 0$. Since $u \in \mathbb{U}$ is arbitrary, $\rho_{\mathbb{U}}(-\zeta) \leq 0$, or

max { $\rho_{U_i}(-\zeta), \rho_{U_i}(-\zeta)$ } ≤ 0.

be such that

max { $\rho_{U_i}(-\zeta), \rho_{U_i}(-\zeta)$ } ≤ 0.

be such that

max { ρ Letting $\varepsilon \to 0^+$, we have $\langle \zeta, -u \rangle \le 0$. Since $u \in \mathbb{U}$ is arbitrary, $\rho_U(-\zeta) \le 0$, or
 $\max{\{\rho_{U_i}(-\zeta), \rho_{U_i}(-\zeta)\}} \le 0$.

Now, let $\zeta \in \hat{N}_{\Omega}(x_0)$ be such that
 $\max{\{\rho_{U_i}(-\zeta), \rho_{U_i}(-\zeta)\}} \le 0$.

We shall **EVALUATE:**

The same that $\max \{\rho_{U_1}(-\zeta), \rho_{U_2}(-\zeta)\}\leq 0$.

We shall prove that $\zeta \in \hat{\partial}^{\alpha} T_{i\ell,12}(\chi_0)$. Assume to the contrary that $\zeta \notin \hat{\partial}^{\alpha} T_{i\ell,12}(\chi_0)$

hen, there exist $C > 0$ and sequences $\{y_i\}$ Frow, i.e. $\xi \in \mathbb{F}_{\Omega}(x_0)$ be such that
 $\max \{ \rho_{U_1}(-\zeta), \rho_{U_2}(-\zeta) \} \le 0.$

We shall prove that $\zeta \in \hat{\mathcal{C}}^{\infty}T_{U_1,\Omega}(x_0)$. Assume to the contrary that $\zeta \notin \hat{\mathcal{C}}^{\infty}T_{U_1}$

Then, there exist $C > 0$ an in $\max \{\rho_{U_1}(-\zeta), \rho_{U_2}(-\zeta)\} \leq 0$.

shall prove that $\zeta \in \hat{\partial}^{\infty}T_{U_4,\Omega}(x_0)$. Assume to the contrary that $\zeta \notin \hat{\partial}^{\infty}T$

re exist $C > 0$ and sequences $\{y_i\} \subset X$, $\{\beta_i\} \subset \mathbb{R}$ such that y_i .

nd β_i $\max \{\rho_{U_1}(-\zeta), \rho_{U_2}(-\zeta)\}\leq 0.$
 $\in \hat{\sigma}^{\infty}T_{\mathcal{U},\Omega}(x_0)$. Assume to the contrary that $\zeta \notin \hat{\sigma}^{\infty}T_{\mathcal{U},\Omega}(x_0)$

and sequences $\{y_i\} \subset X$, $\{\beta_i\} \subset \mathbb{R}$ such that $y_i \to x_0$ as
 $\neq x_0$ and
 $y_i - x_0 \parallel$ max { $\rho_{U_1}(-\zeta)$, $\rho_{U_2}(-\zeta) \le 0$.
 $\iota \zeta \in \hat{\partial}^{\alpha} T_{\mathcal{U},\Omega}(x_0)$. Assume to the contrary that $\zeta \notin \hat{\partial}^{\alpha} T_{\mathcal{U},\Omega}(x_0)$

0 and sequences $\{y_i\} \subset X$, $\{\beta_i\} \subset \mathbb{R}$ such that $y_i \to x_0$ as
 γ , $y_i \ne x$ t C > 0 and sequences { y_i } ⊂ X', { β_i } ⊂ ℝ such that $y_i \rightarrow x_0$ as
 $\geq T_{\mathcal{U},\Omega}(y_i), y_i \neq x_0$ and
 $-x_0$ $\geq C(||y_i - x_0|| + \beta_i)$. (2.4)

We have from (2.4) that

($||y_i - x_0|| + T_{\mathcal{U},\Omega}(y_i)$), $\forall i$. (2.5)
 $t_i := T_{\mathcal{U},\Omega$

$$
\langle \zeta, y_i - x_0 \rangle \ge C(||y_i - x_0|| + \beta_i). \tag{2.4}
$$

for all i . We have from (2.4) that

$$
\langle \zeta, y_i - x_0 \rangle \ge C(||y_i - x_0|| + T_{\mathcal{U}, \Omega}(y_i)), \qquad \forall i.
$$
\nThis yields

This yields

$$
t_i := T_{\mathcal{U},\Omega}(y_i) \le \frac{1}{C} ||\zeta|| ||y_i - x_0||, \quad \forall i
$$

and thus $t_i \rightarrow 0$ as $i \rightarrow \infty$.

 $\begin{aligned} x_0 \text{ and }\\ x_0 \parallel + \beta_i \text{)}. \end{aligned}$

(2.4)

(4) that

(2.5)

(2.5)
 (2.4)

(2.5)

(2.5)
 $t_1^i, t_2^i \ge 0, w^i \in \Omega,$
 ℓ .
 $\zeta \in \hat{N}_{\Omega}(x_0)$, for *i*

(2.6) Ne have from (2.4) that
 $T(\parallel y_i - x_0 \parallel + T_{\mathcal{U},\Omega}(y_i)),$ $\forall i.$ (2.5)

s
 $t_i := T_{\mathcal{U},\Omega}(y_i) \leq \frac{1}{C} || \zeta || \parallel y_i - x_0 ||, \forall i$
 \Rightarrow 0 as $i \rightarrow \infty.$

. By the definition of $T_{\mathcal{U},\Omega}$, for each *i*, there exist $t'_1, t'_2 \geq 0$ by $||||y_i - x_0||$, $\forall i$

for each *i*, there exist $t_1^i, t_2^i \ge 0$, $w^i \in \Omega$,
 $w^j = y_i + t_1^i u_1^i + t_2^i u_2^i$
 $x_0 ||\le ||y_i - x_0|| + (t_i + \eta)M$.
 $\rightarrow \infty$. Let $\varepsilon > 0$. Since $\zeta \in \hat{N}_{\Omega}(x_0)$, for *i*

(2.6)
 \Rightarrow \Rightarrow $\langle \z$ *i*, there exist $t_1^i, t_2^i \ge 0$, $w^i \in \Omega$,
 $t_1^i u_1^i + t_2^i u_2^i$
 $-x_0 \parallel + (t_i + \eta)M$.
 $t \in \mathcal{S} > 0$. Since $\zeta \in \hat{N}_{\Omega}(x_0)$, for *i*

(2.6)
 $\int_0^t -t_1^i u_1^i - t_2^i u_2^i - x_0$
 $\int_0^t u_1^i + t_2^i (-\zeta, u_2^i)$
 $\$ $\frac{1}{C} \|\zeta\| \| y_i - x_0 \|, \quad \forall i$

(ii) for each i, there exist $t'_1, t'_2 \ge 0, w' \in \Omega$,
 $\eta, w' = y_i + t'_1 u'_1 + t'_2 u'_2$
 $\mu'_2 - x_0 \| \le \| y_i - x_0 \| + (t_i + \eta)M.$
 $\le i \to \infty$. Let $\varepsilon > 0$. Since $\zeta \in \hat{N}_{\Omega}(x_0)$, for i
 \vert ,
 $\langle 2.6 \rangle$

$$
t_i < t'_1 + t'_2 < t_i + \eta
$$
, $w^j = y_i + t'_1 u'_1 + t'_2 u'_2$

One has for all i that

$$
|w^{i}-x_{0}||=||y_{i}+t_{1}^{i}u_{1}^{i}+t_{2}^{i}u_{2}^{i}-x_{0}||\leq ||y_{i}-x_{0}||+(t_{i}+\eta)M.
$$

Since $\eta > 0$ is arbitrary, $w^i \to x_0$ as $i \to \infty$. Let $\varepsilon > 0$. Since $\zeta \in \hat{N}_{\Omega}(x_0)$, for i large enough, we have

$$
\langle \zeta, w^i - x_0 \rangle \le \varepsilon \parallel w^i - x_0 \parallel,
$$
\n(2.6)

For *i* large enough,

$$
t_i := T_{\mathcal{U},\Omega}(y_i) \leq \frac{1}{C} || \zeta || ||y_i - x_0||, \quad \forall i
$$

\n $t_i \to 0$ as $i \to \infty$.
\n $t_i \to 0$ as $i \to \infty$.
\n t_j such that
\n $t_i < t_1' + t_2' < t_i + \eta, \quad w^i = y_i + t_1'u_1' + t_2'u_2'$
\nfor all *i* that
\n $||w^i - x_0|| = ||y_i + t_1'u_1' + t_2'u_2' - x_0|| \leq ||y_i - x_0|| + (t_i + \eta)M$.
\n $t_j < 0$ is arbitrary, $w^i \to x_0$ as $i \to \infty$. Let $\varepsilon > 0$. Since $\zeta \in \hat{N}_{\Omega}(x_0)$, for *i* we have
\n $\langle \zeta, w^i - x_0 \rangle \leq \varepsilon || w^i - x_0 ||,$
\n $\langle \zeta, w^i - x_0 \rangle \leq \varepsilon || w^i - x_0 ||,$
\nge enough,
\n $C(||y_i - x_0|| + t_i) \leq \langle \zeta, y_i - x_0 \rangle = \langle \zeta, w^i - t_1'u_1' - t_2'u_2' - x_0 \rangle$
\n $= \langle \zeta, w^i - x_0 \rangle + t_1' \langle -\zeta, u_1' \rangle + t_2' \langle -\zeta, u_2' \rangle$
\n $\leq \varepsilon || w^i - x_0 ||$ (as $\rho_{U_i}(-\zeta) \leq 0$).

Hence,

$$
C \leq \varepsilon \frac{\|w^i - x_0\|}{\|y_i - x_0\| + t_i},
$$
\n(2.7)

Hong Duc University Journal of Science, E7, vol.12, p.(57-65), 2022
 $\|\vec{w} - x_0\|$
 $y_i - x_0\| + t_i$,

'e claim that the sequence $\left\{\frac{\|\vec{w} - x_0\|}{\|y_i - x_0\| + t_i}\right\}$ is bounded. Hong Duc University Journal of Science, E7, vol.12, p.(57-65), 2022
 $\|\vec{w} - x_0\|$ (2.7)
 $\|\vec{y}_i - x_0\| + t_i$,

We claim that the sequence $\left\{\frac{\|\vec{w}^i - x_0\|}{\|\vec{y}_i - x_0\| + t_i}\right\}$ is bounded.

he sequence is not bounded Hong Duc University Journal of Science, E7, vol.12, p.(57-65), 2022
 $\frac{w^j - x_0 \parallel}{\vert x_i - x_0 \vert \vert + t_i},$ (2.7)

claim that the sequence $\left\{ \frac{\Vert w^j - x_0 \Vert}{\Vert y_i - x_0 \Vert + t_i} \right\}$ is bounded.

sequence is not bounded. Then, withou Hong Duc University Journal of Science, E7, vol.12, p.(57-65), 2022
 $C \le \varepsilon \frac{\|\mathbf{w}^j - x_0\|}{\|\mathbf{y}_i - x_0\| + t_i},$ (2.7)

ugh. We claim that the sequence $\left\{\frac{\|\mathbf{w}^i - x_0\|}{\|\mathbf{y}_i - x_0\| + t_i}\right\}$ is bounded. Hong Duc University Journal of Science, E7, vol.12, p.(57-65), 2022
 $\|\vec{w}^j - x_0\|$
 $\|\vec{w}^j - x_0\| + t_i$,

e claim that the sequence $\left\{\frac{\|\vec{w}^j - x_0\|}{\|\vec{y}_i - x_0\| + t_i}\right\}$ is bounded.

e sequence is not bounded. The Hong Duc University Journal of Science, E7, vol.12, p.(57-65), 2022
 $\frac{\sqrt{1-x_0}\|}{-x_0\|+t_i}$, (2.7)

claim that the sequence $\left\{\frac{\|\vec{w}^i - x_0\|}{\|\vec{y}_i - x_0\|+t_i}\right\}$ is bounded. for *i* large enough. We claim that the sequence $\left\{\frac{\Vert W - x_0 \Vert}{\Vert W - x_0 \Vert}\right\}$ is bounded. $0 \parallel \cdot \cdot \cdot_i$ vience, E7, vol.12, p.(57-65), 2022

(2.7)
 $\left\| w^{i} - x_{0} \right\|$
 $y_{i} - x_{0} \left\| + t_{i} \right\}$ is bounded.

without loss of generality, Science, E7, vol.12, p.(57-65), 2022

(2.7)
 $\left\| \frac{w^i - x_0}{v_i - x_0} \right\|_{\infty}$ is bounded.

In, without loss of generality,

(2.8) $i \neq \parallel \parallel$ mce, E7, vol.12, p.(57-65), 2022

(2.7)
 $\frac{w^i - x_0 ||}{\left| \frac{1}{i} - x_0 \right| + t_i}$ is bounded.

without loss of generality,

(2.8) nce, E7, vol.12, p.(57-65), 2022

(2.7)
 $w^{i} - x_0 ||$
 $\left.\frac{1}{x_i - x_0} + t_i\right\}$ is bounded.

vithout loss of generality, ience, E7, vol.12, p.(57-65), 2022

(2.7)
 $\left\| \frac{w^i - x_0 \parallel}{y_i - x_0 \parallel + t_i} \right\}$ is bounded.

without loss of generality,

(2.8) f Science, E7, vol.12, p.(57-65), 2022

(2.7)
 $\left\{\frac{||w^i - x_0||}{||y_i - x_0|| + t_i}\right\}$ is bounded.

en, without loss of generality, f Science, E7, vol.12, p.(57-65), 2022

(2.7)
 $\left\{\frac{\|\vec{w}^i - x_0\|}{\|\vec{y}_i - x_0\| + t_i}\right\}$ is bounded.

en, without loss of generality, f Science, E7, vol.12, p.(57-65), 2022

(2.7)
 $\left\{\frac{\|\mathbf{w}^i - x_0\|}{\|\mathbf{y}_i - x_0\| + t_i}\right\}$ is bounded.

en, without loss of generality,

(2.8) is bounded. Hong Due University Journal of Science, E7, vol.12, p.(57-68), 2022
 $C \leq \varepsilon \frac{\|w' - x_0\|}{\|y_i - x_0\| + t_i}$, (2.7)

ge enough. We claim that the sequence $\frac{\|w' - x_0\|}{\|y_i - x_0\| + t_i}$ is bounded.

ontrary that the sequence arge enough. We claim that the sequence $\left\{\frac{\|w'-x_0\|}{\|y_i-x_0\|+t_i}\right\}$ is bounded.

contrary that the sequence is not bounded. Then, without loss of gener
 $\frac{\|w'-x_0\|}{\|y_i-x_0\|+t_i} > M+2$.

for $i > i_0$,
 $(M+2)(\|y_i-x_0\|$

Assume to the contrary that the sequence is not bounded. Then, without loss of generality, there exists i_0 such that for $i > i_0$, we have

i i , we have 0 0 || || 2. || || i i i w x ^M y x t [−] − + + (2.8) 0 0 (2)(|| ||) || || M y x t y x t M + − + − + i i i i (1) || || 2 0 M y x t + − − i i || || sup . || || i i w x ^Q y x t [−] ⁼ − +

That is, for $i > i_0$,

$$
(M+2)(||y_i-x_0||+t_i) \leq ||w^i-x_0|| \leq ||y_i-x_0||+(t_i+\varepsilon)M
$$

Let $\varepsilon \to 0^+$, one has

$$
(M+2)(||y_i - x_0|| + t_i) \le ||y_i - x_0|| + t_i M
$$

for all i sufficiently large. This implies that

$$
(M+1) \| y_{i} - x_{0} \| \leq -2t_{i} < 0
$$

for all $i > i_0$ large enough. This is a contradiction. Set

$$
Q = \sup_{i} \left\{ \frac{\|w^{i} - x_{0}\|}{\|y_{i} - x_{0}\| + t_{i}} \right\}.
$$

From $\|w' - x_0\| + t$, $\sqrt{2\pi}$ (2.8)

That is, for $i > i_0$,
 $(M + 2)(\|y_i - x_0\| + t_i) \le \|w' - x_0\| \le \|y_i - x_0\| + (t_i + \varepsilon)M$

Let $\varepsilon \to 0^+$, one has
 $(M + 2)(\|y_i - x_0\| + t_i) \le \|y_i - x_0\| + t_iM$

for all *i* sufficiently large. This imp From (2.7), we have $C \leq \varepsilon Q$. Let $\varepsilon \to 0^+$, one gets $C \leq 0$ which is a contradiction. That is, for $i > i_0$,
 $||y_i - x_0||+t_i > M + 2$.

That is, for $i > i_0$,
 $(M + 2)(||y_i - x_0||+t_i) \le ||w' - x_0|| \le ||y_i - x_0|| + (t_i + \varepsilon)M$

Let $\varepsilon \to 0^*$, one has
 $(M + 2)(||y_i - x_0||+t_i) \le ||y_i - x_0||+t_iM$

for all *i* sufficiently large. This im $\frac{w}{||y_i - x_0|| + t_i} > M + 2.$

That is, for $i > i_0$,
 $(M + 2)(||y_i - x_0|| + t_i) \le ||w^i - x_0|| \le ||y_i - x_0|| + (t_i + \varepsilon)M$
 \therefore $(M + 2)(||y_i - x_0|| + t_i) \le ||y_i - x_0|| + t_iM$

or all *i* sufficiently large. This implies that
 $(M + 1)||y_i - x_0|| \le -2t_i < 0$

o $\in \hat{\partial}^{\infty} T_{\mathcal{U}, \Omega}(x_0)$. This ends the proof. $x_i - x_0 ||+t_i \le ||w' - x_0|| \le ||y_i - x_0|| + (t_i + \varepsilon)M$
 $+ 2)(||y_i - x_0||+t_i) \le ||y_i - x_0||+t_iM$

arge. This implies that
 $(M+1)||y_i - x_0|| \le -2t_i < 0$

ough. This is a contradiction.
 $Q = \sup_{i} \left\{ \frac{||w' - x_0||}{||y_i - x_0||+t_i} \right\}$.
 $C \le \varepsilon Q$. Let is a contradiction.

differentials of the

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 ι . If $\zeta \in \hat{N}_{\mathcal{R}(r)}(x_0)$

such that

(2.9) () contradiction.

Exercise the polynomials of $\zeta \in \hat{N}_{\mathcal{R}(r)}(x_0)$
 ζ and ζ and ζ and ζ and ζ and ζ and ζ an

Next, we give the formula for computing Fréchet singular subdifferentials of the minimal time function at a point outside the target. For that aim, we need the following result.

Proposition 2.1. Let $x_0 \in X$ be such that $0 < r := T_{\mathcal{U}, \Omega}(x_0) < +\infty$. If $\zeta \in \hat{N}_{\mathcal{R}(r)}(x_0)$

Let
$$
\varepsilon \to 0^+
$$
, one has
\n
$$
(M+2)(\|y_i - x_0\| + t_i) \le \|y_i - x_0\| + t_iM
$$
\nfor all *i* sufficiently large. This implies that
\n
$$
(M+1) \|y_i - x_0\| \le -2t_i < 0
$$
\nfor all $i > i_0$ large enough. This is a contradiction.
\nSet
\n
$$
Q = \sup_{i} \left\{ \frac{\|w^i - x_0\|}{\|y_i - x_0\| + t_i} \right\}.
$$
\nFrom (2.7), we have $C \le \varepsilon Q$. Let $\varepsilon \to 0^+$, one gets $C \le 0$ which is a contradiction.
\nThus, $\zeta \in \partial^{\infty} T_{\mathcal{U},\Omega}(x_0)$. This ends the proof.
\nNext, we give the formula for computing Fréchet singular subdifferentials of the
\nminimal time function at a point outside the target. For that aim, we need the following result.
\n**Proposition 2.1.** Let $x_0 \in X$ be such that $0 < r := T_{\mathcal{U},\Omega}(x_0) < +\infty$. If $\zeta \in \hat{N}_{\mathcal{R}(r)}(x_0)$
\nthen we have $\rho_U(-\zeta) \ge 0$.
\n*Proof.* Since $\zeta \in \hat{N}_{\mathcal{R}(r)}(x_0)$, for any $\varepsilon > 0$, there exists $\delta > 0$ such that
\n $\langle \zeta, y - x_0 \rangle \le \varepsilon ||y - x_0||$.
\nfor all $y \in B(x_0, \delta) \cap \mathcal{R}(r)$.
\nSince $r = T_{\mathcal{U},\Omega}(x_0) < +\infty$, by the definition of $T_{\mathcal{U},\Omega}$, for $0 < y < r/2$, there exist
\n $t_1, t_2 \ge 0$, $w \in \Omega$, $u_i \in U_1$, $u_i \in U_2$, such that $r < t_1 + t_2 < r + \gamma$ and $w = x_0 + t_1u_1 + t_2u_2$.

for all $y \in B(x_0, \delta) \cap \mathcal{R}(r)$.

for all $i > i_0$ large enough. This is a contradiction.

Set
 $Q = \sup_i \left\{ \frac{\|w' - x_0\|}{\|y_i - x_0\| + t_i} \right\}.$

From (2.7), we have $C \leq \varepsilon Q$. Let $\varepsilon \to 0^+$, one gets $C \leq 0$ which is a contradiction.
 $\zeta \in \hat{\partial}^{\infty} T_{i\ell$ Since $r = T_{\mathcal{U},\Omega}(x_0) < +\infty$, by the definition of $T_{\mathcal{U},\Omega}$, for $0 < \gamma < r/2$, there exist *C* ≤ 0 which is a contradiction.

singular subdifferentials of the

m, we need the following result.
 $f(x_0)(x_0) < +\infty$. If $\zeta \in \hat{N}_{\mathcal{R}(r)}(x_0)$

sists $\delta > 0$ such that

(2.9)

, for $0 < \gamma < r/2$, there exist
 $r + \gamma$ $Q = \sup_{i} \left\{ \frac{||w' - x_0||}{||y_i - x_0|| + t_i} \right\}.$

From (2.7), we have $C \le \varepsilon Q$. Let $\varepsilon \to 0^+$, one gets $C \le 0$ which is a contradiction.

Thus, $\zeta \in \hat{\sigma}^{\infty} T_{i\ell,\Omega}(x_0)$. This ends the proof.

Next, we give the formula Without loss of generality, we may assume that $t_1 \ge t_2$ and $t_1 > 0$. Then, $t_1 > r/2$. a contradiction.

ferentials of the

following result.
 $f \zeta \in \hat{N}_{\mathcal{R}(r)}(x_0)$
 \Rightarrow th that

(2.9)
 $\frac{1}{2}$, there exist
 $\therefore x_0 + t_1 u_1 + t_2 u_2$.
 $t_1 > r/2$.

61

Fréchet singular subdifferentials of the minimal time function associated with a collection of sets
We take $r/2 < \eta < \max\{r, \delta/M\}$ and let $z = x_0 + \eta u_1$. It is easy to see that
 (x_0, δ) . Moreover, since
 $z + (t_1 - \eta)u_1 + t_2u$ $z \in B(x_0, \delta)$. Moreover, since Fréchet singular subdifferentials of the minimal time function associated with a collisting that $r/2 < \eta < \max\{r, \delta/M\}$ and let $z = x_0 + \eta u_1$. It is easy to $z \in B(x_0, \delta)$. Moreover, since $z + (t_1 - \eta)u_1 + t_2u_2 = w \in \Omega$, we ha First remains of the minimal time function associated with a collection of sets
 $X\{r, \delta/M\}$ and let $z = x_0 + \eta u_1$. It is easy to see that
 $z + (t_1 - \eta)u_1 + t_2u_2 = w ∈ Ω$,
 $\Omega(z) ≤ t_1 - \eta + t_2 < r + \gamma - \eta ≤ r$.
 $\Omega, \delta) ∩ R(r)$. Fro differentials of the minimal time function associated with a collection of sets
 $\max\{r, \delta/M\}$ and let $z = x_0 + \eta u_1$. It is easy to see that
 $z + (t_1 - \eta)u_1 + t_2u_2 = w \in \Omega$,
 $T_{\mathcal{U},\Omega}(z) \le t_1 - \eta + t_2 < r + \gamma - \eta \le r$.
 $B(x_0, \$ entials of the minimal time function associated with a collection of sets

{ $r, δ/M$ } and let $z = x_0 + \eta u_1$. It is easy to see that

+ $(t_1 - \eta)u_1 + t_2u_2 = w \in \Omega$,
 $(z) \le t_1 - \eta + t_2 < r + \gamma - \eta \le r$.
 $δ) \cap R(r)$. From (2.9), one h $\label{eq:2.1} \begin{split} \text{Fr\'echet singular subdifferentials of the minimal time function associated with a collection of sets} \\ \text{We take} \quad & r \, / \, 2 < \eta < \max \{r, \delta / M \} \quad \text{and} \quad \text{let} \quad z = x_0 + \eta u_1. \quad \text{It is easy to see that} \\ z \in B(x_0, \delta). \text{ Moreover, since} \\ z + (t_1 - \eta) u_1 + t_2 u_2 = w \in \Omega, \\ \text{we have} \\ T_{t,\Omega}(z) \leq t_1 - \eta + t_2 < r + \gamma - \eta \leq r. \\ \text{It means that} \quad & z \in B(x_0,$ s of the minimal time function associated with a collection of sets
 $\forall M$ } and let $z = x_0 + \eta u_1$. It is easy to see that
 $-\eta)u_1 + t_2u_2 = w \in \Omega$,
 $\forall t_1 - \eta + t_2 < r + \gamma - \eta \le r$.
 $\forall \mathcal{R}(r)$. From (2.9), one has $\langle \zeta, \eta u_1$ Fréchet singular subdifferentials of the minimal time function associated with a collection of

We take $r/2 < \eta < \max\{r, \delta/M\}$ and let $z = x_0 + \eta u_1$. It is easy to see
 (x_0, δ) . Moreover, since
 $z + (t_1 - \eta)u_1 + t_2u_2 = w \in \$ Formal interpretation associated with a collection of sets
 $\max \{r, \delta/M\}$ and let $z = x_0 + \eta u_1$. It is easy to see that
 $z + (t_1 - \eta)u_1 + t_2u_2 = w \in \Omega$,
 $\sum_{U,\Omega}(z) \le t_1 - \eta + t_2 < r + \gamma - \eta \le r$.
 $(x_0, \delta) \cap \mathcal{R}(r)$. From (2.9), of a ssociated with a collection of sets
 $x_0 + \eta u_1$. It is easy to see that

Ω,
 $\eta \le r$.

(b), one has $\langle \zeta, \eta u_1 \rangle \le \varepsilon || \eta u_1 ||$.
 $-\zeta$) ≥ 0. This ends the proof.

(example $x_0 \in X$ such that d with a collection of sets

is easy to see that
 $\langle \zeta, \eta u_1 \rangle \le \varepsilon || \eta u_1 ||$.

is ends the proof.
 $x_0 \in X$ such that
 $x_0 \in S > 0$ such that
 $x_0 \in S > 0$ such that Fréchet singular subdifferentials of the minimal time function associated with a collection of s

We take $r/2 < \eta < \max\{r, \delta/M\}$ and let $z = x_0 + \eta u_1$. It is easy to see th
 $z \in B(x_0, \delta)$. Moreover, since
 $z + (t_1 - \eta)u_1 + t_$

$$
z + (t_1 - \eta)u_1 + t_2 u_2 = w \in \Omega
$$

we have

$$
T_{\mathcal{U},\Omega}(z) \le t_1 - \eta + t_2 < r + \gamma - \eta \le r.
$$

It means that $z \in B(x_0, \delta) \cap \mathcal{R}(r)$. From (2.9), one has $\langle \zeta, \eta u_1 \rangle \leq \varepsilon ||\eta u_1||$.

$$
\langle \zeta, u_1 \rangle \leq \varepsilon ||u_1||.
$$

Theorem 2.2. Let U_1 and U_2 be convex and $x_0 \in X$ such that

$$
\hat{\partial}^{\infty} T_{\mathcal{U},\Omega}(x_0) = \hat{N}_{\mathcal{R}(r)}(x_0) \cap \{\zeta \in X^* : \max\{\rho_{U_1}(-\zeta), \rho_{U_2}(-\zeta)\} = 0\}.
$$
 (2.10)

 $\langle \eta \rangle \langle \max \{r, \delta/M\} \rangle$ and let $z = x_0 + \eta u_1$. It is easy to see that

er, since
 $z + (t_1 - \eta)u_1 + t_2u_2 = w \in \Omega$,
 $T_{U,\Omega}(z) \le t_1 - \eta + t_2 \langle r + \gamma - \eta \le r$.
 $z \in B(x_0, \delta) \cap \mathcal{R}(r)$. From (2.9), one has $\langle \zeta, \eta u_1 \rangle \le \varepsilon || \eta u_$ ke $r/2 < \eta < \max\{r, \delta/M\}$ and let $z = x_0 + \eta u_1$. It is easy to see that

. Moreover, since
 $z + (t_1 - \eta)u_1 + t_2u_2 = w \in \Omega$,

we
 $T_{U,\Omega}(z) \le t_1 - \eta + t_2 < r + \gamma - \eta \le r$.

ans that $z \in B(x_0, \delta) \cap \mathcal{R}(r)$. From (2.9), one has $\langle \zeta$ Frequencial solution is solution associated with a concernance set of
 ϵ take $r/2 < \eta < \max\{r, \delta/M\}$ and let $z = x_0 + \eta u_1$. It is easy to see that
 δ). Moreover, since
 $z + (t_1 - \eta)u_1 + t_2u_2 = w \in \Omega$,

have
 $T_{\lambda t,\Omega}(z)$ Freeders suggarit subtributions of the minimizar direction associated with a conceduratives
 $r/2 < \eta < \max\{r, \delta/M\}$ and let $z = x_0 + \eta u_1$. It is easy to see that
 $z + (t_1 - \eta)u_1 + t_2u_2 = w \in \Omega$,
 x_0, δ). Moreover, since
 Proof. Assume that $\zeta \in \hat{\partial}^{\infty} T_{\mathcal{U}, \Omega}$. Then, for any $\varepsilon > 0$, there exists $\delta > 0$ such that

 $\epsilon B(x_0, \delta)$. Moreover, since
 $\epsilon B(x_0, \delta)$. Moreover, since
 $\tau(t_1 - \eta)u_1 + t_2u_2 = w \epsilon \Omega$,

we have
 $T_{\delta(\Omega)}(z) \le t_1 - \eta + t_2 < r + \gamma - \eta \le r$.

It means that $z \in B(x_0, \delta) \cap \mathcal{R}(r)$. From (2.9), one has $\langle \zeta, \eta u_1 \rangle \le \varepsilon ||\eta$ $\in B(x_0, \delta)$. Moreover, since
 $z + (t_1 - \eta)u_1 + t_2u_2 = w \in \Omega$,

we have
 $T_{i,j,\Omega}(z) \le t_1 - \eta + t_2 < r + \gamma - \eta \le r$.

It means that $z \in B(x_0, \delta) \cap \mathcal{R}(r)$. From (2.9), one has $\langle \zeta, \eta u_1 \rangle \le \varepsilon ||\eta u_1||$.

quivalently,
 $\langle \zeta, u$ $z + (t_1 - \eta)u_1 + t_2u_2 = w \in \Omega,$

we have
 $T_{\{i,\Omega}(z) \le t_1 - \eta + t_2 < r + \gamma - \eta \le r.$

It means that $z \in B(x_0, \delta) \cap \mathcal{R}(r)$. From (2.9), one has $\langle \zeta, \eta u_1 \rangle \le \varepsilon || \eta u_1 ||$.

Let $\varepsilon \to 0 +$, we get $\langle \zeta, u_1 \rangle \le 0$. Therefore, $\zeta \in \hat{N}_{\mathcal{R}(r)}(x_0)$.

Example 12 and $T_{\mathcal{U},\Omega}(z) \le t_1 - \eta + t_2 < r + \gamma - \eta \le r$.

means that $z \in B(x_0, \delta) \cap \mathcal{R}(r)$. From (2.9), one has $\langle \zeta, \eta u_1 \rangle \le$

ently,
 $\langle \zeta, u_1 \rangle \le \varepsilon ||u_1||$.

t $\varepsilon \to 0 +$, we get $\langle \zeta, u_1 \rangle \le 0$. Therefore, ρ we have
 $T_{U,\Omega}(z) \leq t_1 - \eta + t_2 < r + \gamma - \eta \leq r$.

It means that $z \in B(x_0, \delta) \cap \mathcal{R}(r)$. From (2.9), one has $\langle \zeta, \eta u_1 \rangle \leq \varepsilon || \eta u_1 ||$.

Equivalently,

Let $\varepsilon \to 0 +$, we get $\langle \zeta, u_1 \rangle \leq \varepsilon ||u_1 ||$.

Let $\varepsilon \to 0 +$, Since $r = T_{U,\Omega}(x_0) < +\infty$, for $0 < \gamma < r^2/4$, there exist $t_1, t_2 \ge 0$, $w \in \Omega$, $u_1 \in U_1$, $\begin{split} T_{i\ell,\Omega}(z) \leq & t_1 - \eta + t_2 < r + \gamma - \eta \leq r. \\ \text{It means that} \quad z \in B(x_0,\delta) \cap \mathcal{R}(r) \quad \text{From} \quad (2.9), \text{ one has } \langle \zeta, \eta u_i \rangle \leq \varepsilon \|\eta u_i\|. \\ \text{Equivalently,} \\ \text{Let } & \varepsilon \to 0+ \text{, we get } \langle \zeta, u_i \rangle \leq 0 \text{ . Therefore, } \rho_{\ell}(-\zeta) \geq 0 \text{ . This ends the proof. } \\ \text{Theorem 2.2. Let } & U_1 \quad \text{$ $(-\zeta) \ge 0$. This ends the proof.

vex and $x_0 \in X$ such that
 $(x_0), \rho_{U_2}(-\zeta) = 0$. (2.10)
 >0 , there exists $\delta > 0$ such that
 $(x,\beta) \in \text{epi}(T_{U,\Omega})$. (2.11)
 $y \in \mathcal{R}(r) \cap B(x_0, \delta)$, that is,
 $y \in \text{exist } t_1, t_2 \ge 0$,

 U_1 , one has $w \in x_0 - \lambda u + \lambda U_1 + t_1 U_1 + t_2 U_2 = x_0 - \lambda u + (t_1 + \lambda) U_1 + t_2 U_2$.

Let $\varepsilon \to 0 +$, we get $\langle \zeta, u_t \rangle \ge \varepsilon ||u_t||$.

Let $\varepsilon \to 0 +$, we get $\langle \zeta, u_t \rangle \le 0$. Therefore, $\rho_U(-\zeta) \ge 0$. This ends the proof.

Theorem 2.2. Let U_1 and U_2 be convex and $x_0 \in X$ such that
 $\varepsilon r := T_{i\ell,\Omega}(x$ (+ + μ + μ Theorem 2.2. Let U_1 and U_2 be convex and $X_0 \in X$ such that
 $\hat{\sigma}^c T_{i,\Omega}(x_0) < \infty$. Then,
 $\hat{\sigma}^c T_{i,\Omega}(x_0) = \hat{N}_{\hat{X}(r)}(x_0) \cap \{\xi \in X^* : \max\{\rho_{U_1}(-\zeta), \rho_{U_2}(-\zeta)\} = 0\}$. (2.10)

Proof. Assume that $\zeta \in \hat{\sigma}^$

Let $\gamma \to 0^+$, we get $\lambda \langle -\zeta, u \rangle \leq \lambda \varepsilon (\|u\| + 1)$.

 \therefore Then,
 \Rightarrow $\hat{\mathcal{R}}_{\mathcal{R}(r)}(x_0) \cap \{\zeta \in X^* : \max{\{\rho_{U_1}(-\zeta), \rho_{U_2}(-\zeta')\}} = 0\}.$ (2.10)

me that $\zeta \in \frac{\delta^{\omega}}{L_{\mathcal{R},\Omega}}$. Then, for any $\varepsilon > 0$, there exists $\delta > 0$ such that
 $y - x_0 || + |\beta - r|$, $\forall y \in B(x_0, \delta$ Divide both sides of the latter inequality by $\lambda > 0$ and then let $\varepsilon \to 0^+$, we obtain *Proof.* Assume that $\zeta \in \hat{\sigma}^{\alpha}T_{U,\Omega}$. Then, for any $\varepsilon > 0$, there exists $\delta > \langle \zeta, y - x_0 \rangle \le \varepsilon (\|y - x_0\| + |\beta - r|)$, $\forall y \in B(x_0, \delta), (y, \beta) \in \text{epi}(T_{U,\Omega})$.

It follows that $\langle \zeta, y - x_0 \rangle \le \varepsilon \|y - x_0\|, \forall y \in \mathcal{R}(r) \$ $\langle \epsilon \partial^{\alpha} T_{\lambda L\Omega}$. Then, for any $\varepsilon > 0$, there exists $\delta > 0$ such that $|\beta - r|$, $\forall y \in B(x_0, \delta), (y, \beta) \in \text{epi}(T_{\lambda L\Omega})$. (2.11)
 $\langle \zeta, y - x_0 \rangle \leq \varepsilon ||y - x_0||, \forall y \in \mathcal{R}(r) \cap B(x_0, \delta)$, that is,
 $\Rightarrow \infty$, for $0 < y < r^2$

 $\langle \zeta', y-x_0 \rangle \leq \varepsilon(\|y-x_0\| + |\beta - r|), \quad \forall y \in B(x_0, \delta), (y, \beta) \in \text{epi}(T_{\mathcal{U}, \Omega}).$ (2.11)

It follows that $\langle \zeta', y-x_0 \rangle \leq \varepsilon \|y-x_0\|, \forall y \in \mathcal{R}(r) \cap B(x_0, \delta),$ that is,
 $\zeta \in \hat{N}_{\mathcal{R}(r)}(x_0).$ Since $r = T_{\mathcal{U}, \Omega}(x_0) \iff \text{for$ Conversely, let $\zeta \in \hat{N}_{\mathcal{R}(r)}(x_0)$ be such that $\rho_U(-\zeta) = 0$. We show that ($\langle \zeta, y - x_0 \rangle \le \varepsilon || y - x_0 ||, \forall y \in \mathcal{R}(r) \cap B(x_0, \delta)$, that is,
 $\langle \langle \zeta, y - x_0 \rangle \rangle \le \varepsilon || y - x_0 ||, \forall y \in \mathcal{R}(r) \cap B(x_0, \delta)$, that is,
 $\langle \zeta, y - x_0 \rangle$, for $0 \le y \le r^2/4$, there exist $t_1, t_2 \ge 0$, $w \in \Omega$, $u_1 \in U_1$,
 (x_0).

ce $r = T_{\mathcal{U},\Omega}(x_0) < +\infty$, for $0 < \gamma < r^2/4$, there exist $t_1, t_2 \ge 0$, $w \in \Omega$,

auch that $r < t_1 + t_2 < r + \gamma$, $w = x_0 + t_1u_1 + t_2u_2$.
 $u \in \mathbb{U}$ be arbitrary and $\lambda > 0$. Assume that $u \in U_1$. Then, by the co $\zeta \in N_{\mathcal{R}(r)}(x_0)$.

Since $r = T_{\mathcal{U},\Omega}(x_0) < +\infty$, for $0 < \gamma < r^2/4$, there exist $t_1,t_2 \ge 0$, $w \in \Omega$, $u_2 \in U_2$ such that $r < t_1 + t_2 < r + \gamma$, $w = x_0 + t_1u_1 + t_2u_2$.

Let $u \in \mathbb{U}$ be arbitrary and $\lambda > 0$. Assume $\epsilon \partial^{\infty} T_{\mathcal{U},\Omega}(x_0)$. Assume to the contrary that $\zeta \notin \partial^{\infty} T_{\mathcal{U},\Omega}(x_0)$. Then, there exists $\alpha > 0$ $T_{\ell,\Omega}(x_0) < +\infty$, for $0 < \gamma < r^2/4$, there exist $t_1, t_2 \ge 0$, $w \in \Omega$, $u_1 \in U_1$,

at $r < t_1 + t_2 < r + \gamma$, $w = x_0 + t_1u_1 + t_2u_2$.

I be arbitrary and $\lambda > 0$. Assume that $u \in U_1$. Then, by the convexity of
 $x_0 - \lambda u + \lambda U$ ²/4, there exist $t_1, t_2 \ge 0$, $w \in \Omega$, $u_1 \in U_1$,
 $t_1u_1 + t_2u_2$.

ume that $u \in U_1$. Then, by the convexity of
 $x_0 - \lambda u + (t_1 + \lambda)U_1 + t_2U_2$.
 $+ \lambda$. For λ sufficiently small, we have
 $u \ge \varepsilon (\|\neg u\| + |\lambda + \gamma|)$.
 Since $r = T_{\ell_1\Omega}(x_0) < +\infty$, for $0 < y < r^2/4$, there exist $t_1, t_2 \ge 0$, $w \in \Omega$, $u_1 \in U_1$,
 $u_2 \in U_2$ such that $r < t_1 + t_2 < r + \gamma$, $w = x_0 + t_1u_1 + t_2u_2$.

Let $u \in \mathbb{U}$ be arbitrary and $\lambda > 0$. Assume that $u \in U_1$ /4, there exist $t_1, t_2 \ge 0$, $w \in \Omega$, $u_1 \in U_1$,
 $\frac{1}{1}u_1 + t_2u_2$.

ne that $u \in U_1$. Then, by the convexity of
 $0 - \lambda u + (t_1 + \lambda)U_1 + t_2U_2$.

λ. For λ sufficiently small, we have
 $\leq \varepsilon(||-\lambda u|| + |\lambda + \gamma|)$.

-1).

y exist $t_1, t_2 \ge 0$, $w \in \Omega$, $u_1 \in U_1$,
 $\in U_1$. Then, by the convexity of
 $\frac{1}{1} + \lambda U_1 + t_2 U_2$.
 λ sufficiently small, we have
 $u \parallel + |\lambda + \gamma|$).

and then let $\varepsilon \to 0^+$, we obtain
 $t \langle -\zeta, u \rangle \le 0$. Since $u \in U$ $\mathcal{L}_2 \geq 0$, $w \in \Omega$, $u_1 \in U_1$,

then, by the convexity of
 $+t_2U_2$.

iently small, we have
 $+\gamma$).

let $\varepsilon \to 0^+$, we obtain
 $\gamma \leq 0$. Since $u \in \mathcal{U}$ is
 $\rho_U(-\zeta) = 0$.
 $= 0$. We show that

en, there exis and sequences $\{y_i\} \subset X$, $\{\beta_i\} \subset \mathbb{R}$ such that $y_i \to x_0$, $y_i \neq x_0$, $r_i := T_{\mathcal{U},\Omega}(y_i) \leq \beta_i$ and 2 ∈ *U*₂ such that $r < t_1 + t_2 < r + \gamma$, $w = x_0 + t_1u_1 + t_2u_2$.

Let $u \in U$ be arbitrary and $\lambda > 0$. Assume that $u \in U_1$. Then, by the convexity of λ_1 , one has $w \in x_0 - \lambda u + \lambda U_1 + t_1U_1 + t_2U_2 = x_0 - \lambda u + (t_1 + \lambda)U_1 + t_2$

We consider two cases:

Hong Due University Journal of Science, E7, vol.12, p.(57-65), 2022

We consider two cases:

Case 1. There exists a subsequence of $\{y_i\}$ which we still denote by $\{y_i\}$ such
 $T_{U,\Omega}(y_i) \leq T_{U,\Omega}(x_0)$ for all i. In th **Hong Duc University Journal of Science, E7, vol.12, p.(57-65), 2022**
We consider two cases:
Case 1. There exists a subsequence of $\{y_i\}$ which we still denote by $\{y_i\}$ such
that $T_{\mathcal{U},\Omega}(y_i) \leq T_{\mathcal{U},\Omega}(x_0)$ fo that $T_{\mathcal{U},\Omega}(y_i) \leq T_{\mathcal{U},\Omega}(x_0)$ for all i. In this case, $y_i \in \mathcal{R}(r)$ for all i. Since $\zeta \in \hat{N}_{\mathcal{R}(r)}(x_0)$ 2, p.(57-65), 2022

by $\{y_i\}$ such
 $\zeta \in \hat{N}_{\mathcal{R}(r)}(x_0)$, for any $\varepsilon > 0$, Hong Duc University Journal of Science, E7, vol.12, p.(57-65), 2022

bsequence of $\{y_i\}$ which we still denote by $\{y_i\}$ such
 i. In this case, $y_i \in \mathcal{R}(r)$ for all *i*. Since $\zeta \in \hat{N}_{\mathcal{R}(r)}(x_0)$
 $\langle \zeta, y_i -$ **Hong Duc University Journal of Science, E7, vol.12, p.(57-65), 2022**

two cases:

e exists a subsequence of $\{y_i\}$ which we still denote by $\{y_i\}$ such
 $(\alpha(x_0)$ for all *i*. In this case, $y_i \in \mathcal{R}(r)$ for all *i*. **Hong Due University Journal of Science, E7, vol.12, p.(57-65), 2022**

We consider two cases:

Case 1. There exists a subsequence of $\{y_i\}$ which we still denote by $\{y_i\}$ such
 $T_{U,\Omega}(y_i) \le T_{U,\Omega}(x_0)$ for all i. In t **Example 2.** Hong Duc University Journal of Science, E7, vol.12, p.(57-65), 2022

We consider two cases:

Case 1. There exists a subsequence of $\{y_i\}$ which we still denote by $\{y_i\}$ such
 $\begin{aligned}\n\langle \mathcal{C}_k, y_i - x_0 \rangle &\le$ **Hong Due University Journal of Science, E7, vol.12, p.(57-65), 2022**

We consider two cases:
 $Case \ I$. There exists a subsequence of $\{y_i\}$ which we still denote by $\{y_i\}$ such

that $T_{\mathcal{U},\Omega}(y_i) \leq T_{\mathcal{U},\Omega}(x_0)$ iversity Journal of Science, E7, vol.12, p.(57-65), 2022

{ y_i } which we still denote by { y_i } such
 $c, y_i \in \mathcal{R}(r)$ for all *i*. Since $\zeta \in \hat{N}_{\mathcal{R}(r)}(x_0)$
 $\mathcal{E} || y_i - x_0 ||$

.12), one gets
 $+ | \beta_i - r | \ge \alpha || y_i - x_0 ||$ Hong Due University Journal of Science, E7, vol.12, p.(57-65), 2022

ics:

i.es:

a subsequence of $\{y_i\}$ which we still denote by $\{y_i\}$ such

or all *i*. In this case, $y_i \in \mathcal{R}(r)$ for all *i*. Since $\zeta \in \hat{N}_{\math$ Hong Due University Journal of Science, E7, vol.12, p.(57-65), 2022

S:

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s:

subsequence of {*y_i*} which we still denote by {*y_i*} such

all *i*. In this case, $y_i \in \mathcal{R}(r)$ for all *i*. Since $\zeta \in \hat{N}_{\mathcal{R}(r$

for *i* large enough. Combining with (2.12) , one gets

$$
\varepsilon \|\, y_i - x_0\,\|\ge \alpha(\|\, y_i - x_0\,\| + |\,\beta_i - r\,|) \ge \alpha \|\, y_i - x_0\,\|
$$

that $T_{\mathcal{U},\Omega}(y_i) > T_{\mathcal{U},\Omega}(x_0)$ for all i. By (2.12), $r_i = T_{\mathcal{U},\Omega}(y_i) < +\infty$ and

$$
0 < r_i - r \leq \beta_i - r \leq \frac{1}{\alpha} \|\zeta\| \|\mathbf{y}_i - \mathbf{x}_0\|
$$

We consider two cases:

Case 1. There exists a subsequence of $\{y_i\}$ which we still denote by $\{y_i\}$ such
 $\{x_{i\Omega}(y_i) \le T_{i\Omega_i}(x_0)$ for all *i*. In this case, $y_i \in \mathcal{R}(r)$ for all *i*. Since $\zeta \in \hat{N}_{\mathcal{R}(r)}(x_$ We consider two cases:

Case 1. There exists a subsequence of $\{y_i\}$ which we still denote by $\{y_i\}$ such

that $T_{i,l,\Omega}(y_i) \le T_{i,l,\Omega}(x_0)$ for all i. In this case, $y_i \in \mathcal{R}(r)$ for all i. Since $\zeta \in \hat{N}_{\mathcal{R}(r)}(x_$ if y_i , there exists a statistiquence of y_i y_i which we still denote by y_i ; such $\varepsilon > 0$,
 $\langle \zeta, y_i - x_0 \rangle \le \varepsilon ||y_i - x_0||$

i large enough. Combining with (2.12), one gets
 $\varepsilon ||y_i - x_0||$

i large enough. Combining if ϵ is a subsequence of $\{y_i\}$ which we sun denote by $\{y_i\}$ such $\langle \zeta, 0 \rangle$ for all i. In this case, $y_i \in \mathcal{R}(r)$ for all i. Since $\zeta \in \hat{N}_{\mathcal{R}(r)}(x_0)$
 $\langle \zeta, y_i - x_0 \rangle \le \varepsilon ||y_i - x_0||$

enough. Combining r all *i*. In this case, $y_i \in \mathcal{R}(r)$ for all *i*. Since $\zeta \in N_{\mathcal{R}(r)}(x_0)$
 $\langle \zeta, y_i - x_0 \rangle \le \varepsilon || y_i - x_0 ||$

Combining with (2.12), one gets
 $v_0 || \ge \alpha(||y_i - x_0|| + | \beta_i - r ||) \ge \alpha || y_i - x_0 ||$

This is a contradiction.

a subs ightarrow of the interpretation.
 $\langle \zeta, y_i - x_0 \rangle \leq \varepsilon ||y_i - x_0||$
 $\langle \zeta, y_i - x_0 \rangle \leq \varepsilon ||y_i - x_0||$

gh. Combining with (2.12), one gets
 $y_i - x_0 ||\geq \alpha (||y_i - x_0|| + |B_i - r|) \geq \alpha ||y_i - x_0||$
 $\leq \varepsilon$. This is a contradiction.
 $\langle \zeta, y_i - x_0 \rangle \le \varepsilon ||y_i - x_0||$
for *i* large enough. Combining with (2.12), one gets
 $\varepsilon ||y_i - x_0|| \ge \alpha(||y_i - x_0|| + |\beta_i - r|) \ge \alpha ||y_i - x_0||$
which implies $\alpha \le \varepsilon$. This is a contradiction.
Case 2. There exists a subsequence of $\langle \zeta, y_i - x_0 \rangle \leq \varepsilon ||y_i - x_0||$

the moral in (2.12), one gets
 $\|\geq \alpha(\|y_i - x_0\| + |\beta_i - r|) \geq \alpha \|y_i - x_0\|$

this is a contradiction.

subsequence of $\{y_i\}$ which is still denoted by $\{y_i\}$ such
 $\|i \cdot By (2.12), r_i = T_{U_i}\Omega(y_i$ 1, y_i , x_{ij} , x_{ij} , x_{ij} and $(2,12)$, one gets
 $\|f(x_i) = x_0\| + |B_i - r| \ge \alpha \|y_i - x_0\|$

is a contradiction.

is a contradiction.

By (2.12), $r_i = T_{U_i \Omega_i}(y_i) < +\infty$ and

By (2.12), $r_i = T_{U_i \Omega_i}(y_i) < +\infty$ and
 $r \le \beta_i - r \le$ in the substract of $|z_1|^2$, one gets
 $|\geq \alpha(||y_i - x_0|| + |\beta_i - r|) \geq \alpha ||y_i - x_0||$

his is a contradiction.

subsequence of $\{y_i\}$ which is still denoted by $\{y_i\}$ such
 $1 i. By (2.12), r_i = T_{U,\Omega}(y_i) < +\infty$ and
 $r_i - r \leq \beta_i - r \leq \frac$ mbining with (2.12), one gets
 $||\ge \alpha(||y_i - x_0|| + |\beta_i - r|) \ge \alpha ||y_i - x_0||$

This is a contradiction.

Subsequence of $\{y_i\}$ which is still denoted by $\{y_i\}$ such
 $||1 \ i. By (2.12), r_i = T_{\mathcal{U},\Omega}(y_i) < +\infty$ and
 $||r_i - r \le \beta_i - r \le \frac{1}{$ $\varepsilon \|\, |y_i - x_0\| \ge \alpha(\|\, y_i - x_0\| + |\beta_i - r|) \ge \alpha \|\, |y_i - x_0\|$

which implies $\alpha \le \varepsilon$. This is a contradiction.

Case 2. There exists a subsequence of $\{y_i\}$ which is still denoted by $\{y_i\}$ such
 $C_A(x_i)(y_i) > T_{i,i,0}(x_0)$ This is a contradiction.

subsequence of $\{y_i\}$ which is still denoted by $\{y_i\}$ such

all i . By (2.12), $r_i = T_{i\ell,\Omega}(y_i) < +\infty$ and
 $\langle r_i - r \leq \beta_i - r \leq \frac{1}{\alpha} || \zeta || || y_i - x_0 ||$
 $\langle r_i \rightarrow r$ as $i \rightarrow \infty$. Hence, we may assu Case 2. There exists a sloosequence of $\{y_i\}$ which is sun denoted by $\{y_i\}$ such $\langle x_{i,0}(y_i) \rangle > T_{i,i,0}(x_0)$ for all i . By $(2.12), r_i = T_{i,i,0}(y_i) < +\infty$ and
 $0 < r_i - r \le \beta_i - r \le \frac{1}{\alpha} ||\xi|| ||y_i - x_0||$

for all i . It impl ϵ × + + ϵ + + ϵ $\begin{aligned}\n\mathbf{y}_i - r &\leq \beta_i - r \leq \frac{1}{\alpha} \|\zeta\| \|\mathbf{y}_i - \mathbf{x}_0\| \\
\mathbf{y}_i - r &\leq \frac{1}{\alpha} \|\zeta\| \|\mathbf{y}_i - \mathbf{x}_0\| \\
\mathbf{y}_i - r &\leq \alpha.\end{aligned}$ Hence, we may assume that $\mathbf{S}(r_i - r) < r$ the minimal time function, for each *i*, there exi i.e. and $i \mapsto g(x) = \frac{1}{2} \left\| \left\langle f \right\| \right\| y_i - x_0 \right\|$
 $0 < r_i - r \leq \beta_i - r \leq \frac{1}{\alpha} \left\| \left\langle f \right\| \right\| y_i - x_0 \right\|$

s that $r_i \to r$ as $i \to \infty$. Hence, we may assume that $5(r_i - r) < r$

ion of the minimal time function, for ea implies that $r_i \rightarrow r_i \rightarrow r_i \rightarrow \alpha$ (if θ) with $r_i \rightarrow r$ as $i \rightarrow \infty$. Hence, we may assume that $S(r_i - r) < r$
definition of the minimal time function, for each i, there exist $t'_1, t'_2 \ge 0$,
 $u'_2 \in U_2$ such that
 $r_i < t^i - 1 + t'_$ $0 < r_i - r \leq \beta_i - r \leq \frac{1}{\alpha} ||\zeta|| ||y_i - x_0||$

mplies that $r_i \to r$ as $i \to \infty$. Hence, we may assume that $5(r_i - r) < r$

efinition of the minimal time function, for each *i*, there exist $t'_1, t'_2 \geq 0$,
 $u'_2 \in U_2$ such that
 r for all *i*. It implies that $r_i \rightarrow r$ as $i \rightarrow \infty$. Hence, we may assume that $5(r_i - r) \le r$
 i. By the definition of the minimal time function, for each *i*, there exist $t'_1, t'_2 \ge 0$,
 $2, u'_i \in U_1, u'_2 \in U_2$ such that
 $r_i \$ $i \rightarrow \infty$. Hence, we may assume that $5(r_i - r) < r$

al time function, for each *i*, there exist $t'_1, t'_2 \ge 0$,
 $-r$, $w_i = y_i + t'_1 u'_1 + t'_2 u'_2$.

assume that $t'_1 \ge t'_2$. Then, for all *i*,
 t'_2) $> \frac{1}{2} r_i > 3(r_i - r)$.
 $\forall r$). Th

$$
r_i < t^i - 1 + t_2^i < 2r_i - r, \quad w_i = y_i + t_1^i u_1^i + t_2^i u_2^i.
$$

$$
t_1^i \geq \frac{1}{2} (t_1^i + t_2^i) > \frac{1}{2} r_i > 3(r_i - r).
$$

This means that $y_i + \gamma_i u_i^i \in \mathcal{R}(r)$. Moreover,

$$
|| y_i + y_i'u_i - x_0 || \le || y_i - x_0 || + 3(r_i - r)M \to 0 \text{ as } i \to \infty.
$$

By the definition of the find
find the finding tend on, for each t, there exist
 $u_1^i \in U_1$, $u_2^i \in U_2$ such that
 $r_i < t^i - 1 + t_2^i < 2r_i - r$, $w_i = y_i + t'_i u_1^i + t'_2 u_2^i$.

Thout loss of generality, we may assume that $t'_1 \$ $\hat{V}_{R(r)}(x_0)$, for *i* sufficiently large, for all *i*. By the definition of the minimal time function, for each *i*, there exist *i*
 $w_i \in \Omega$, $u'_1 \in U_1$, $u'_2 \in U_2$ such that
 $r_i < t^i - 1 + t'_2 < 2r_i - r$, $w_i = y_i + t'_1 u'_1 + t'_2 u'_2$.

Without loss of generality, we may as uch that
 $-1+t'_2 < 2r_i - r$, $w_i = y_i + t'_i u'_i + t'_2 u'_2$.

ality, we may assume that $t'_i \ge t'_j$. Then, for all *i*,
 $t'_i \ge \frac{1}{2}(t'_i + t'_2) > \frac{1}{2}r_i > 3(t_i - r)$.
 $t'_i - 2r, 3r_i - 3r$). Then,
 $w_i = y_i + γ_i u'_i + (t'_i - γ_i)u'_i + t'_2 u'_2$.
 $\le t'_$ i such that
 $\varepsilon t^i - 1 + t_2^i < 2r_i - r$, $w_i = y_i + t_1^i u_1^i + t_2^i u_2^i$.

cerality, we may assume that $t_1^i \ge t_2^i$. Then, for all i ,
 $t_1^i \ge \frac{1}{2} (t_1^i + t_2^i) > \frac{1}{2} r_i > 3(r_i - r)$.
 $(2r_i - 2r, 3r_i - 3r)$. Then,
 $w_i = y_i$ $r_i \le t^i - 1 + t^i_2 \le 2r_i - r$, $w_i = y_i + t^i_i u^i_1 + t^i_2 u^i_2$.

Without loss of generality, we may assume that $t^i_1 \ge t^i_2$. Then, for all i ,
 $t^i_i \ge \frac{1}{2}(t^i_1 + t^i_2) > \frac{1}{2}t^i_i > 3(t_i - r)$.

For each i , let $\gamma_i \in (2r_i$ it loss of generality, we may assume that $t'_1 \ge t'_2$. Then, for all i,
 $t'_1 \ge \frac{1}{2}(t'_1 + t'_2) > \frac{1}{2}r'_1 > 3(r'_1 - r)$.

ch i, let $\gamma_i \in (2r_i - 2r, 3r_i - 3r)$. Then,
 $w_i = y_i + \gamma_i u'_i + (t'_i - \gamma_i) u'_i + t'_2 u'_2$.
 $T_{i,j,\Omega}(y_i + \gamma_i u'_i) \le t'_1$ 1ch i, let $\gamma_i \in (2r_i - 2r_33r_i - 3r)$. Then,
 $w_i = y_i + \gamma_i u_i^i + (t_i^i - \gamma_i) u_i^i + t_2^i u_2^i$.
 $T_{\ell(\Omega)}(y_i + \gamma_i u_i^i) \le t_i^i - \gamma_i + t_2^i < 2r_i - r - (2r_i - 2r) = r$.

means that $y_i + \gamma_i u_i^i \in \mathcal{R}(r)$. Moreover,
 $||y_i + \gamma_i^i u_i - x_0|| \le ||y_i - x_$ cach *i*, let $\gamma_i \in (2r_i - 2r, 3r_i - 3r)$. Then,
 $w_i = y_i + \gamma \mu_i' + (t_1' - \gamma_i)\mu_i' + t_2'\mu_2'.$

s, $T_{i,(i)}(y_i + \gamma_i\mu_i') \le t_1' - \gamma_i + t_2' < 2r_i - r - (2r_i - 2r) = r$.

s means that $y_i + \gamma_i\mu_i' \in \mathcal{R}(r)$. Moreover,
 $||y_i + \gamma_i'\mu_i' - x_0|| \le ||y_i - x_0$

$$
\langle \zeta, y_i + \gamma_1^i u_i - x_0 \rangle \leq \varepsilon ||y_i + \gamma_1^i u_i - x_0||.
$$

$$
\langle \zeta, y_i - x_0 \rangle \leq \varepsilon \| y_i + \gamma_i u_1^i - x_0 \| + \gamma_i \langle -\zeta, u_1^i \rangle \leq \varepsilon \| y_i + \gamma_i u_1^i - x_0 \|.
$$

Combining with (2.12), one has

$$
\alpha(||y_i - x_0|| + |r_i - r|) \leq \alpha(||y_i - x_0|| + |B_i - r|) \leq \varepsilon ||y_i + y_i u_1^i - x_0||.
$$

Thus,

Fréchet singular subdifferentials of the minimal time function associated with a collection of sets
\nis,
\n
$$
\alpha \leq \varepsilon \frac{||y_i + y_i u_i^i - x_0||}{||y_i - x_0|| + |r_i - r|}
$$
\n
$$
\leq \varepsilon \frac{||y_i - x_0|| + 3M |r_i - r|}{||y_i - x_0|| + |r_i - r|}
$$
\n
$$
\leq (3M + 1)\varepsilon.
$$
\nring $\varepsilon \to 0^+$, we have $\alpha \leq 0$. This is a contradiction. Therefore,
\n $\Omega(x_0)$. The proof is complete.
\n*adjment:* This research was supported by Hong Duc University under grant
\nT-2020-01.

e minimal time function associated with a collection of sets
 $+ \gamma_i u_i^i - x_0 ||$
 $+ \gamma_i u_i^i - x_0 ||$
 $\frac{x_0 || + |r_i - r|}{|x_i - x_0|| + |r_i - r|}$
 $\frac{1}{\alpha} \leq 0$. This is a contradiction. Therefore,
 $\alpha \leq 0$. This is a contradiction. T the minimal time function associated with a collection of sets
 $y_i + \gamma_i u_i^i - x_0 ||$
 $-x_0 || + |r_i - r ||$
 $-x_0 || + 3M |r_i - r ||$
 $y_i - x_0 || + |r_i - r ||$
 $+ 1) \varepsilon$.
 $\alpha \le 0$. This is a contradiction. Therefore, <u>minimal time function associated with a collection of sets

+ $\gamma_i u_1^i - x_0 ||$
 $x_0 || + |r_i - r||$
 $x_0 || + 3M |r_i - r||$
 $-x_0 || + |r_i - r||$
 $)\varepsilon$.</u>
 $x \le 0$. This is a contradiction. Therefore, Letting $\varepsilon \to 0^+$, we have $\alpha \le 0$. This is a contradiction. Therefore, Fréchet singular subdifferentials of the minimal time function associated with a collection

Thus,
 $\alpha \leq \varepsilon \frac{||y_i + \gamma_i u_i^i - x_0||}{||y_i - x_0|| + |r_i - r|}$
 $\leq \varepsilon \frac{||y_i - x_0|| + |\gamma_i - r|}{||y_i - x_0|| + |r_i - r|}$

Letting $\varepsilon \to 0^*$, we ha $\in \hat{\partial}^{\infty} T_{\mathcal{U},\Omega}(x_0)$. The proof is complete.

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