INTEGRATED RESOLVENT OPERATORS AND NONDENSELY INTEGRODIFFERENTIAL EQUATIONS INVOLVING THE NONLOCAL CONDITIONS

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Abstract: *The aim of this work is to prove some results of the existence and regularity of solutions for some nondensely integrodifferential equations with nonlocal conditions, where the linear part has an integrated resolvent operator in the sens given by Oka [7]. They extend the results of [4] and [5].*

Keywords: *Integrated resolvent operator, resolvent operator, integral solution, nonlocal, nondensely, integrodifferential equations.*

1. Introduction

Nonlocal conditions in dynamical systems play an important role in many physical problems. They have better effects in applications than the classical initial conditions $u(0)=u_0$. See, for example, in [1,2] to determine the unknown physical parameter in some inverse heat condition problems and in [3] to describe the diffusion phenomenon of a small amount of gas in a transparent tube. As indicated in [8], we sometimes need to deal with non-densely defined operators. For example, when we look at a one-dimensional heat equation with Dirichlet conditions on $[0,\pi]$ and consider $A=\frac{\partial^2}{\partial x^2}$ in $C([0,\pi],\mathbb{R})$, in order to measure the solutions in the sup-norm, then the domain.

$$\mathrm{D}(A)=\left\{u\in C^2([0,\pi],\mathbb{R}):u(0)=u(\pi)=0\right\}$$

is not dense in $C([0,\pi],\mathbb{R})$ with the sup-norm since

$$\overline{\mathcal{D}(A)}=\left\{u\in C([0,\pi],\mathbb{R}):u(0)=u(\pi)=0\right\}\neq C([0,\pi],\mathbb{R}).$$

In this work, we are concerned with the existence and regularity of solutions for the following nondensely nonlocal integrodifferential equation

$$u'(t)=Au(t)+\int_0^t B(t-s)u(s)ds+f(t,u(t)) \qquad \text{for } t\in[0,a] \tag{1.1}$$
$$u(0)=u_0+g(u)$$

where $A:D(A)\subset X\to X$ is a nondensely defined closed linear operator on a Banach space X, $(B(t))_{t\geq 0}$ is a family of closed linear operators on X having the same domain $D(B)\supset D(A)$ which is independent of t, $f:[0,a]\times X\to X$ and $g:C([0,a];X)\to X$ are given functions to be specified later, where $C([0,a];X)$ denotes the space of continuous function form $[0,a]$ to X.

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In [5], Ezzinbi and Liu studied the special case of (1.1) when $B(t) \equiv 0 \ \forall t \in [0,a]$. More precisely, they studied the following nonlocal evolution equations

$$\begin{aligned} &\frac{du}{dt} = Au(t) + f(t,u(t)), \ t \in [0,1] \\ &u(0) = u_0 + g(u), \end{aligned} \tag{1.2}$$

By using the fixed-point methods and the theory of integrated semigroup Ezzinbi and Liu obainted the existence and uniqueness of mild solution to (1.2) when A is not necessarily densely defined but satisfies the Hille-Yosida condition. Then, they verified that mild solutions are ``strict solutions" if additional conditions are assumed.

It is worth emphasizing that in [4] Ezzinbi and Ghnimi proved the existence and regulariy of solutions to (1.1) when A is densely defined and has a resolvent operator in the sens given by Grimmer [6]. However, in the case that the operator A is not densely defined, their results the existence and regulariy are not guaranteed.

2. Integrated resolvent operators

In this section, we summarize basic results which are useful in the sequel. Let Z and W be Banach spaces. We denote by $\mathcal{L}(Z,W)$ the Banach space of bounded linear operators from Z into W endowed with the operator norm, and we abbreviate to $\mathcal{L}(Z)$ when $Z = W$.

Let $A : D(A) \subseteq X \to X$ be a closed linear operator whose domain is not necessarily densely defined in X and $(B(t))_{t\geq 0}$ be a family of linear operators in X with $D(A) \subset D(B(t))$ for $t \geq 0$ and of bounded linear operators from Y into X. Here, Y is the Banach space $D(A)$ equipped with the graph norm $|y|_Y := |Ay| + |y|$ for $y \in Y$. We start by putting together the fundamental properties on integrated resolvent operators. We refer to Oka [7] for more details. Let us consider the following integrodifferential equation:

$$\begin{cases} x'(t) = Ax(t) + \int_0^t B(t-s)x(s)\mathrm{d}s & \text{for } t \geq 0 \\ x(0) = x_0 \in X \end{cases} \tag{2.1}$$

Definition 2.1. ([7]) An integrated resolvent operator for Eq.(2.1) is a bounded operator-valued function $R(t) \in \mathcal{L}(X)$ for $t \geq 0$, having the following properties.

r_1 : For all $x \in X, R(.)x \in C([0,+\infty); X)$.

r_2 : For all $x \in X, \int_0^{\cdot} R(s)xds \in C([0,+\infty); Y)$.

r_3 : $R(t)x - tx = A\int_0^t R(s)x\,ds + \int_0^t B(t-s)\int_0^s R(r)xdrds$ for all $x \in X$ and $t \geq 0$.

r_4 : $R(t)x - tx = \int_0^t R(s)Axds + \int_0^t \int_0^s R(s-r)B(r)xdrds$ for all $x \in D(A)$ and $t \geq 0$.

Remark 2.2. Definition 2.1 generalizes that of integrated semigroup of A when $B=0$

Definition 2.3. An integrated resolvent operator $(R(t))_{t\geq 0}$ in $\mathcal{L}(X)$ is called locally Lipschitz continuous, if for all $a>0$, there exists a constant $C_a = C(a) > 0$ such that:

$$|R(t)-R(s)| \leq C_a |t-s| \text{ for } t,s \in [0,a].$$

Theorem 2.4. ([7]) Suppose that $(R(t))_{t\geq 0}$ is a locally Lipschitz continuous integrated resolvent operator. Then for all $x \in \overline{D(A)}, t \to R(t)x$ is a C^1-function on $[0,+\infty)$.

We now introduce the following assumptions:

(H0): The operator A satisfies the Hille-Yosida condition.

(H1): $(B(t))_{t\geq 0}$ is a family linear operator in X with $D(A) \subset D(B(t))$ for all $t \geq 0$ and, of bounded linear operators from Y to X such that the functions $B(.)x$ are of strong bounded variation on each finite interval $[0,a], a>0$, for $x \in D(A)$.

The following result provides sufficient conditions ensuring the existence of locally Lipschitz continuous integrated resolvent operator for Eq.(2.1).

Theorem 2.5. ([7]) *Assume that* **(H0)** *and* **(H1)** *hold. Then, there exists a unique locally Lipschitz continuous integrated resolvent operator of Eq.*(2.1).

We study the following initial value problem:

$$\begin{cases} x'(t) = Ax(t) + \int_0^t B(t-s)x(s)ds + q(t) \quad \text{for } t \geq 0 \\ x(0) = x_0 \end{cases} \tag{2.2}$$

where $x_0 \in X$ and $q \in C([0,+\infty);X)$. We shall introduce the notions of integral and strict solutions to Eq.(2.2) and give some results concerning the existence and regularity of solutions of Eq.(2.2) used in the later sections.

Definition 2.6. ([7]) Let $q \in L^1_{\text{loc}}(0,+\infty;X)$ and $x_0 \in X$. A function $x:[0,+\infty) \to X$ is called an integral solution of Eq.(2.2) if the following conditions hold:

i) $x \in C([0,+\infty);X)$.

ii) $\int_0^{.} x(s)ds \in C([0,+\infty);Y)$.

iii) $x(t) = x_0 + A\int_0^t x(s)ds + \int_0^t B(t-s)\int_0^s x(r)drds + \int_0^t q(s)ds$ for $t \geq 0$.

Remark 2.7. If x is an integral solution of Eq. (1.1) then, it follows from Definition 2.6. that $x(t) \in \overline{D(A)}$ for all $t \geq 0$. Indeed, $x(t) = \lim_{h\to 0} \frac{1}{h}\int_t^{t+h} x(s)ds$ and $\int_t^{t+h} x(s)ds \in D(A)$. In particular, $x(0) \in \overline{D(A)}$ is a necessary condition for existence of an integral solution of Eq.(2.2).

Definition 2.8. A function $x:[0,+\infty) \to X$ is called a strict solution of Eq.(2.2) if the following conditions hold:

i) $x \in C^1([0,+\infty);X) \cap C([0,+\infty);Y)$.

ii) x satisfies Eq.(2.2) on $[0,+\infty)$.

Theorem 2.9. ([7]) *Assume that* $\overline{D(A)} = X$ *and* $\rho(A) \neq \varnothing$. Let $q \in L^1_{\text{loc}}(0,+\infty;X)$. *The following statements are equivalent*

i) *Eq.*(2.1) *admits a locally Lipschitz continuous integrated resolvent operator* $(R(t))_{t\geq 0}$.

ii) *Eq.* (2.1) *admits a resolvent operator* $(R(t))_{t\geq 0}$.

iii) *For all* $x_0 \in X$ *there exists a unique integral solution* x *to* Eq.(2.2).

iv) *For all* $x_0 \in X$, *there exists a unique weak solution* x *to Eq.*(2.2). *In this case,*

$$R(t)x_0 = \int_0^t R(s)x_0 ds \text{ for } t \geq 0 \text{ and } x_0 \in X$$

$$x(t) = \frac{d}{dt}\left(R(t)x_0 + \int_0^t R(t-s)q(s)ds\right) \text{ for } t \geq 0$$

$$= R(t)x_0 + \int_0^t R(t-s)q(s)ds \text{ for } t \geq 0 \text{ and } x_0 \in X$$

The following is a key Theorem to prove our main results.

Theorem 2.10 ([7]). *Let a family* $(U(t))_{t\geq 0}$ *in* $\mathcal{L}(X)$ *be locally Lipschitz continuous with* $U(0)=0$. *Then, the following holds:*

i) If $q \in L^1(0,a;X)$, *then* $\int_0^t U(.-s)q(s)ds \in C^1([0,a];X)$.

Putting $Q(t) := \frac{d}{dt}\int_0^t U(t-s)q(s)ds$ *for* $t \in [0,a]$, *we have*

$$|Q(t)| \leq C_a \int_0^t |q(s)|\, ds$$

where C_a *is the Lipschitz constant of* $U(t)$ *on* $[0,a]$. *Moreover, if* $|q(t)| \leq K$ *for* $t \in [0,a]$, *then*

$$|Q(t+s) - Q(t)| \leq KC_a s + C_a \int_0^t |q(s+r) - q(r)|\, dr \text{ for } s,t,t+s \in [0,a].$$

ii) *If a function* $q:[0,a] \to X$ *is of strong bounded variation, the function* $Q(.)$ *defined in* i) *is Lipschitz continuous on* $[0,a]$.

Remark 2.11. The results reported in Theorem 2.10 hold for any locally Lipschitz continuous family of bounded linear operators $(U(t))_{t\geq 0}$ with $U(0)=0$. In particular, these results are true for the integrated resolvent operators.

The following theorem gives sufficient conditions for the existence of integral and strict solutions of Eq.(2.2).

Theorem 2.12. ([7]) *Assume that Eq.*(2.1) *has an integrated resolvent operator* $(R(t))_{t\geq 0}$ *that is locally Lipschitz continuous and* $\rho(A) \neq \varnothing$. *Then, the following holds.*

i) *If* $x_0 \in \overline{D(A)}$ *and* $q \in L^1(0,a;X)$, *then there exists a unique integral solution* $x(.)$ *of Eq.* (2.2) *which is given by the variation of constants formula*

$$x(t) = R^{'}(t)x_0 + \frac{d}{dt}\int_0^t R(t-s)q(s)ds \text{ for } t \in [0,a].$$

Moreover, we have $|x(t)| \le C_a\left(|x_0| + \int_0^t |q(s)|ds\right)$ for $t \in [0,a]$.

ii) *If* $x_0 \in D(A), q \in W^{1,1}(0,a;X)$ *and* $Ax_0 + q(0) \in \overline{D(A)}$, *then there exists a unique strict solution* $x(.)$ *of Eq.*(2.2). *Moreover, we have*

$$\left|x^{'}(t)\right| \le C_a\left(\left|Ax_0 + q(0)\right| + \int_0^t \left|B(s)x_0 + q^{'}(s)\right|ds\right) \quad \text{for } t \in [0,a].$$

3. Existence and Regularity of Solutions

Definition 3.1. A continuous function $u:[0,a] \to X$ is said to be a strict solution of Eq.(1.1) if

i) $u \in C^1([0,a];X) \cap C([0,a];Y)$,

ii) u satisfies Eq.(1.1).

Definition 3.2. A continuous function $u:[0,a] \to X$ is said to be a mild solution of Eq.(1.1) if

$$u(t) = R'(t)\left(u_0 + g(u)\right) + \frac{d}{dt}\int_0^t R(t-s)f(s,u(s))ds \text{ for } t \in [0,a].$$

To prove the existence of mild solutions, we make the following assumptions:

(**H2**): Function $f:[0,a]\times X \to X$ is continuous and Lipschitzian with respect to the second argument. Let $L_f > 0$ be such that

$$|f(t,u) - f(t,v)| \le L_f|u-v| \text{ for } t \in [0,a] \text{ and } u,v \in X.$$

(**H3**): Function $g: C([0,a];X) \to X$ is Lipschitz continuous. Let $L_g > 0$ be such that

$$|g(u) - g(v)| \le L_g|u-v| \text{ for } u,v \in C([0,a];X).$$

Theorem 3.3. *Assume that Eq.* (2.1) *has an integrated resolvent operator* $(R(t))_{t\ge 0}$ *that is locally Lipschitz continuous and* $\rho(A) \ne \emptyset$. *Let* f, g *be two functions satisfying* (**H2**) *and* (**H3**) *respectively, and* $u_0 + g(u) \in \overline{D(A)}$. *Then the nonlocal problem* (1.1) *has a unique mild solution* u *on* $[0,a]$ *provided that*

$$C_a\left(L_g + L_f a\right) < 1. \tag{3.1}$$

Proof. Consider the operator $\Phi: C([0,a];X) \to C([0,a];X)$ defined by

$$(\Phi u)(t) = R'(t)\left[u_0 + g(u)\right] + \frac{d}{dt}\int_0^t R(t-s)f(s,u(s))ds \quad \text{for} \quad t \in [0,a].$$

Let $u, v \in C([0,a];X)$. Then for $t \in [0,a]$, we have

$$|(\Phi u)(t)-(\Phi v)(t)| \le |R'(t)[g(u)-g(v)]| + \left|\frac{d}{dt}\int_0^t |R(t-s)[f(s,u(s))-f(s,v(s))]|\,ds\right|$$

$$\le C_a\left(L_g\,|u-v| + L_f\int_0^t |u(s)-v(s)|\,ds\right)$$

$$\le C_a\left(L_g + L_f a\right)|u-v|,$$

which implies that $|(\Phi u)-(\Phi v)| \le M_a\left(L_g+L_f a\right)|u-v|$.

Thus, from (3.1), Φ is a strict contraction. Then by the Banach's fixed point theorem Φ has a unique fixed point in $C([0,a];X)$, which means there exists a unique mild solution for Equation (1.1) on $[0,a]$. For the regularity of the mild solution, we assume the following assumption:

(H4): $f \in C^1([0,a]\times X;X)$ and the partial derivatives $D_1 f(.,.)$ and $D_2 f(.,.)$ are locally Lipschitzian with respect to the second argument.

Theorem 3.4. *Assume that Equation* (2.1) *has an integrated resolvent operator* $(R(t))_{t\ge 0}$ *that is locally Lipschitz continuous and* $\rho(A)\ne\varnothing$. *Let* (**H2**) - (**H4**) *hold and*

$$u(0)=u_0+g(u)\in D(A) \qquad (3.2)$$
$$Au(0)+f(0,u(0))\in \overline{D(A)}.$$

Then, the integral solution of Equation (1.1) *given by* Theorem 3.3 *is a strict solution on* $[0,+\infty)$.

Proof. Let u be the mild solution of Equation (1.1) given by Theorem 3.3. Then

$$u(t)=R'(t)u(0)+\frac{d}{dt}\int_0^t R(t-s)f(s,u(s))ds \text{ for } t\in[0,a]. \qquad (3.3)$$

Differentiating (r_4) with $x=u(0)\in D(A)$, we get

$$R'(t)u(0)=.(0)+R(t)Au(0)+\int_0^t R(t-s)B(s)u(0)ds \quad \text{ for } t\in[0,a].$$

which implies, together with (3.3), that

$$u(t)=u(0)+R(t)Au(0)+\int_0^t R(t-s)B(s)u(0)ds \qquad (3.4)$$
$$+\frac{d}{dt}\int_0^t R(t-s)f\left(s,u(s)\right)ds \quad \text{ for } t\in[0,a].$$

Consider now the following equation

$$\frac{dv(t)}{dt}=Av(t)+\int_0^t B(t-s)v(s)ds+D_1 f(t,u(t))+D_2 f(t,u(t))v(t) \qquad (3.5)$$
$$+B(t)u(0) \text{ for } t\in[0,a]$$
$$v(0)=A\,u(0)+f\left(0,\,u(0)\right).$$

where D_1 and D_2 are the partial derivatives to the first and second variables, respectively.

Then by the contraction mapping principle we can prove that equation (3.5) has an integral solution v on $[0,T]$ which is given by

$$v(t) = R'(t)\left[A\,u(0) + f\left(0,\, u(0)\right)\right] \tag{3.6}$$
$$+\frac{d}{dt}\int_0^t R(t-s)\left[D_1 f(s,u(s)) + D_2 f(s,u(s))v(s) + B(s)u(0)\right]ds.$$

Let w the function be defined by

$$w(t) = u(0) + \int_0^t v(s)ds \text{ for } t \in [0,a].$$

Now, we shall prove that $w = u$. In view of (3.2) it follows from (3.6) that the solution v of Eq. (3.5) satisfies

$$v(t) = R'(t)\left[A\,u(0) + f\left(0,\, u(0)\right)\right]$$
$$+\frac{d}{dt}\int_0^t R(t-s)\left[D_1 f(s,u(s)) + D_2 f(s,u(s))v(s) + B(s)u(0)\right]ds.$$

Integrating this over $[0,t]$, we obtain

$$\int_0^t v(s)ds = R(t)[Au(0) + f(0,u(0))]$$
$$+\int_0^t R(t-s)\left[D_1 f(s,u(s)) + D_2 f(s,u(s))v(s) + B(s)u(0)\right]ds$$

and so

$$R(t)Au(0) = -R(t)f(0,u(0)) + \int_0^t v(s)ds \tag{3.7}$$
$$-\int_0^t R(t-s)\left[D_1 f(s,u(s)) + D_2 f(s,u(s))v(s) + B(s)u(0)\right]ds.$$

On the other hand, since the function $t \to w(t)$ is continuously differentiable, it follows from Theorem 2.10 that

$$t \to \int_0^t R(t-s)f\left(s, w(s)\right)ds$$

is also continuously differentiable and

$$\frac{d}{dt}\int_0^t R(t-s)f\left(s,w(s)\right)ds = \frac{d}{dt}\int_0^t R(s)f\left(t-s, w(t-s)\right)ds$$
$$= R(t)\,f\left(0,u(0)\right) + \int_0^t R(t-s)\left[D_1 f\left(s,w(s)\right) + D_2 f\left(s,w(s)\right)v(s)\right]ds.$$

This implies that

$$R(t)f(0,u(0)) = \frac{d}{dt}\int_0^t R(t-s)f\left(s,w(s)\right)ds \tag{3.8}$$
$$-\int_0^t R(t-s)\left[D_1 f\left(s,w(s)\right) + D_2 F\left(s,w(s)\right)v(s)\right]ds.$$

Combining (3.4) with (3.7) and (3.8), we find

$$
\begin{aligned}
u(t) &= u(0) - R(t)f(0,u(0)) + \int_0^t v(s)ds \\
&\quad - \int_0^t R(t-s)\left[D_1 f(s,u(s)) + D_2 f(s,u(s))v(s) + B(s)u(0)\right]ds \\
&\quad + \int_0^t R(t-s)B(s)u(0)ds + \frac{d}{dt}\int_0^t R(t-s)f\left(s,u(s)\right)ds \\
&= w(t) - \frac{d}{dt}\int_0^t R(t-s)f\left(s,w(s)\right)ds \\
&\quad + \int_0^t R(t-s)\left[D_1 f\left(s,w(s)\right) + D_2 F\left(s,w(s)\right)v(s)\right]ds \\
&\quad - \int_0^t R(t-s)\left[D_1 f(s,u(s)) + D_2 f(s,u(s))v(s) + B(s)u(0)\right]ds \\
&\quad + \int_0^t R(t-s)B(s)u(0)ds + \frac{d}{dt}\int_0^t R(t-s)f\left(s,u(s)\right)ds \\
&= w(t) + \frac{d}{dt}\int_0^t R(t-s)\left[f(s,u(s)) - f(s,w(s))\right]ds \\
&\quad - \int_0^t R(t-s)\left[D_1 f(s,u(s)) - D_1 f(s,w(s))\right]ds \\
&\quad - \int_0^t R(t-s)\left[D_2 f(s,u(s)) - D_2 f(s,w(s))\right]v(s)ds.
\end{aligned}
$$

Consequently,

$$
\begin{aligned}
u(t) - w(t) &= \frac{d}{dt}\int_0^t R(t-s)\left[f(s,u(s)) - f(s,w(s))\right]ds \\
&\quad - \int_0^t R(t-s)\left[D_1 f(s,u(s)) - D_1 f(s,w(s))\right]ds \\
&\quad - \int_0^t R(t-s)\left[D_2 f(s,u(s)) - D_2 f(s,w(s))\right]v(s)ds
\end{aligned}
$$

and so,

$$
\begin{aligned}
|u(t) - w(t)| &\le \left|\frac{d}{dt}\int_0^t R(t-s)\left[f(s,u(s)) - f(s,w(s))\right]ds\right| \\
&\quad + \left|\int_0^t R(t-s)\left[D_1 f(s,u(s)) - D_1 f(s,w(s))\right]ds\right| \\
&\quad + \left|\int_0^t R(t-s)\left[D_2 f(s,u(s)) - D_2 f(s,w(s))\right]v(s)ds\right|.
\end{aligned}
$$

Let $\mathcal{K}=\{u(t),w(t):t\in[0,a]\}$. Then $\mathcal{K}$ is a compact set. Since $D_1 f$ and $D_2 f$ are locally Lipschitz with respect to the second argument, then $D_1 f$ and $D_2 f$ are globally Lipschitz on $\mathcal{K}$. Thus there exists $\gamma(a)>0$ such that

$$|u(t)-w(t)|\leq \gamma(a)\int_0^t |u(s)-v(s)|\,ds \text{ for } t\in[0,a]$$

where $\gamma(a)=C_a L_f + b_0 \operatorname{Lip}\left(D_1 f\right)+b_0^2 \operatorname{Lip}\left(D_2 f\right)$

with $b_0 := \max\left\{\sup_{0\leq s\leq a}|R(s)|, \sup_{0\leq s\leq a}\left|v(s)\right|\right\}.$

By Gronwall's lemma, we deduce that $u(t)=w(t)$ for $t\in[0,a]$. Then u is continuously differentiable in $[0,a]$. So the function $t\to f(t,u(t))$ is continuously differentiable on v, which means, by Theorem 2.12 that u is a strict solution of Eq.(1.1) on $[0,a]$.

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