SOME FIXED POINT THEOREMS FOR FISHER-TYPE CONTRACTIVE MAPPINGS

Nguyen Huu Hoc[1], Hoang Kim Anh[2], Le Thi Thu[2]

Received: 18 March 2022/ Accepted: 25 March 2022/ Published: April 2022

Abstract: *In this paper, we provide some new fixed point theorems for mappings satisfying Fisher - type contractive conditions in metric spaces. Some examples are also given to illustrate our results.*

Keywords: *Boundedly compact, fixed point property, metric space, T-orbitally compact space.*

1. Itroduction and Preliminaries

In 1976, Fisher [1] introduced and proved several results for mappings satisfying different contractive conditions. One of them is the following fixed point theorem.

Theorem 1.1. ([1]) *If T is a continuous mapping of the compact metric space X into itself satisfying the inequality*

$$\left[d\left(Tx,Ty\right)\right]^2 < d\left(x,Tx\right).d\left(y,Ty\right)+c.d\left(x,Ty\right)d\left(y,Tx\right) \tag{1.1}$$

for all distinct x,y in X, where $0 \le c$, then T is a fixed point mapping. Futher, if $0 \le c \le 1$, then the fixed point of T is unique.

In this article, we call mappings satisfying the condition (1.1) as Fisher-type contractive mappings. It is well-known that every Banach contraction mapping is continuous. Here, Fisher also requires the continuity of the mapping. We consider the relationship between the continuity of mappings and the existence of fixed points, as well as the necessity of the compactness of the underlying spaces.

First of all, we recall some definitions that will be used in this article.

Definition 1.1. A metric space (X,d) is said to be boundedly compact if every bounded sequence in X has a convergent subsequence.

Definition 1.2. Let (X,d) be a metric space and T be a self-mapping on X. The orbit of T at $x \in X$ is defined as: $O_x(T) = \{x, Tx, T^2x, T^3x, \dots\}$

Definition 1.3. Let (X,d) be a metric space and T be a self-mapping on X. Then, X is said to be T-orbitally compact if every sequence in $O_x(T)$ has a convergent subsequence for all $x \in X$.

[1] Faculty of Natural Sciences, Hong Duc University; Email: nguyenhuuhoc@hdu.edu.vn

[2] K22 High Quality Mathematics Teacher Education, Hong Duc University

In ([2]), H. Garai et al. have shown that T-orbital compactness of a space depends on the mapping T defined on it.

Definition 1.4. Let (X,d) be a metric space and $T:X\to X$ be a self mapping. T is said to be orbitally continuous at a point z in X if for any sequence $\{x_n\}\subseteq O_x(T)$ for some $x\in X, x_n\to z$ as $n\to\infty$ implies $Tx_n\to Tz$ as $n\to\infty$.

Definition 1.5. Let (X,d) be a metric space and $T:X\to X$ be a mapping. For $x_0\in X$ the sequence $\{x_n\}$ defined by $x_n=Tx_{n-1}$ for $n\geq 1$ is called Picard iteration sequence with the initial point x_0. The mapping T is said to be a Picard operator if it has a unique fixed point and every Picard iteration in X converges to the fixed point.

2. Main results

Now, we are in a position to state our main results.

Theorem 2.1. *Let (X,d) be a compact metric space and let $T:X\to X$ such that*

$$\left[d\left(Tx,Ty\right)\right]^2<d\left(x,Tx\right).d\left(y,Ty\right)+c.d\left(x,Ty\right)d\left(y,Tx\right)$$

for all $x,y\in X, x\neq y$ and $0\leq c\leq 1$. Then, T has a unique fixed point.

Proof. Set $m=\inf\{d(x,Tx):x\in X\}$,

Then, there exists a sequence $(x_n)\subset X$ such that $\lim\limits_{n\to\infty}d\left(x_n,Tx_n\right)=m,$ and, by the compactness of X, there exists a subsequence (x_{n_k}) of (x_n) such that $x_{n_k}\to u\in X$, $Tx_{n_k}\to w\in X$ as $k\to\infty$. We have $d\left(x_{n_k},Tx_{n_k}\right)\to d\left(u,w\right)=m$ as $k\to\infty$.

If $m>0$ then $u\neq w$.

If $Tx_n=Tw$ for n in an infinite subset I of $\mathbb{N}$, then $Tx_n=Tw\to w$. So w is a fixed point.

When I is a finite set, we denote $(x'_n)=(x_n)\setminus\{x_m:Tx_m=Tw, m\in I\}$.

Assume now that $Tx_n\neq Tw\ \forall n$, we have

$$\left[d\left(Tx_n,Tw\right)\right]^2<d\left(x_n,Tx_n\right)d\left(w,Tw\right)+cd\left(x_n,Tw\right)d\left(w,Tx_n\right),$$

letting $n\to\infty$, we get

$$\left[d\left(w,Tw\right)\right]^2\leq d\left(u,w\right)d\left(w,Tw\right).$$

This implies that

$$d\left(w,Tw\right)\leq d\left(u,w\right)=m.$$

And from

$$\left[d\left(Tw,T^2w\right)\right]^2<d\left(w,Tw\right)d\left(Tw,T^2w\right)+cd\left(w,T^2w\right)d\left(Tw,Tw\right),$$

We obtain $d\left(Tw,T^2w\right)<d\left(w,Tw\right)\le m,$ which is a contradiction. Thus, we must have $m=0$. Therefore, $u=w$.

If $Tx_n \neq Tu$, we have

$$\left[d\left(Tx_n,Tu\right)\right]^2<d\left(x_n,Tx_n\right)d\left(u,Tu\right)+cd\left(x_n,Tu\right)d\left(u,Tx_n\right),$$

Taking the limit as $n\to\infty$, we obtain $\left[d\left(u,Tu\right)\right]^2\le 0.$

This implies that $u=Tu$, i.e., u is a fixed point of T.

Next, we check the uniqueness of u. Arguing by contradiction, let u' be another fixed point of T, then

$$\left[d\left(u,u'\right)\right]^2=\left[d\left(Tu,Tu'\right)\right]^2<d\left(u,Tu\right)d\left(u',Tu'\right)+cd\left(u,Tu'\right)d\left(u',Tu\right)=c\left[d\left(u,u'\right)\right]^2,$$

which is a contradition. The fixed point must therefore be unique. This completes the proof of the theorem.

In Theorem 2.1, the existence of the fixed point of Fisher-type contractive mappings without assuming the continuity of the mapping. However, at fixed point, the mapping is still continuous at the fixed point.

Indeed, let (x_n) be a sequence in X. If $x_n\neq u$, we have

$\left[d\left(Tx_n,Tu\right)\right]^2<d\left(x_n,Tx_n\right)d\left(u,Tu\right)+cd\left(x_n,Tu\right)d\left(u,Tx_n\right)$, which implies that

$$d\left(Tx_n,Tu\right)<cd\left(x_n,u\right). \tag{2.1}$$

By (2.1), if $x_n\to u$ as $n\to\infty$ then $Tx_n\to Tu$ as $n\to\infty$. Thus, T is continuous at the fixed point u.

Theorem 2.2. *Let (X,d) be a boundedly compact metric space and $T:X\to X$ be an orbitally continuous mapping such that*

$$\left[d\left(Tx,Ty\right)\right]^2<d\left(x,Tx\right)d\left(y,Ty\right)+c.d\left(x,Ty\right).d\left(y,Tx\right)$$

for all $x,y\in X$ with $x\neq y$ and $0\le c<1$. Then T is a Picard operator.

Proof. Let $x_0\in X$ be arbitrary but fixed and consider the iterative sequence (x_n), where $x_n=T^nx_0$ for each $n\in\mathbb{N}$. If the sequence (x_n) has two equal consecutive terms, then T must have a fixed point. So, we may assume that no two consecutive terms of (x_n) are equal.

We denote $s_n=d(x_n,x_{n+1})$. Then $s_n>0$ for each $n\in\mathbb{N}$. We have

$$\begin{aligned}
s_n^2&=\left[d\left(T^nx_0,T^{n+1}x_0\right)\right]^2\\
&<d\left(T^{n-1}x_0,T^nx_0\right)d\left(T^nx_0,T^{n+1}x_0\right)+c.d\left(T^{n-1}x_0,T^{n+1}x_0\right)d\left(T^nx_0,T^nx_0\right)\\
&=d\left(T^{n-1}x_0,T^nx_0\right)d\left(T^nx_0,T^{n+1}x_0\right)\\
&=s_{n-1}s_n.
\end{aligned}$$

Thus $s_n < s_{n-1}$. This shows that (s_n) is a strictly decreasing sequence of positive real numbers. Hence, it converges to some $b \geq 0$. For each $n \in \mathbb{N}$, we have

$$s_n < s_{n-1} < \ldots < s_1 = K \text{ (say)}.$$

Thus, for all n, m, one has

$$\begin{aligned} t^2 = \left[d\left(x_n, x_m\right)\right]^2 &= \left[d\left(T^n x_0, T^m x_0\right)\right]^2 \\ &< d\left(T^{n-1}x_0, T^n x_0\right) d\left(T^{m-1}x_0, T^m x_0\right) + c.d\left(T^{n-1}x_0, T^m x_0\right) d\left(T^{m-1}x_0, T^n x_0\right) \\ &< d\left(T^{n-1}x_0, T^n x_0\right) d\left(T^{m-1}x_0, T^m x_0\right) \\ &+ c\left[d\left(T^{m-1}x_0, T^m x_0\right) + d\left(T^m x_0, T^n x_0\right)\right]\left[d\left(T^{n-1}x_0, T^n x_0\right) + d\left(T^n x_0, T^m x_0\right)\right] \\ &= s_{n-1}s_{m-1} + c\left(s_{n-1} + t\right)\left(s_{m-1} + t\right) \\ &< K^2 + c\left(K + t\right)^2. \end{aligned}$$

This implies that $(1-c)t^2 - 2cKt - K^2(1+c) < 0,$ or, $t = d\left(x_n, x_m\right) < K\dfrac{1+c}{1-c}.$

Therefore, (x_n) is a bounded sequence in X. By the bounded compactness property of X, (x_n) must have a convergent subsequence, say (x_{n_k}), which converges to some $z \in X$. By the orbital continuity of T, (Tx_{n_k}) converges to Tz. We have

$$s_{n_k} = d(x_{n_k}, Tx_{n_k}),$$
$$s_{n_k+1} = d(Tx_{n_k}, T^2 x_{n_k}).$$

Taking the limit as $k \to \infty$, we obtain $b = d(z, Tz) = d(Tz, T^2 z).$

We are going to show that $b = 0$.

Assume that $b > 0$. Then, $z \neq Tz$ and

$$\left[d\left(Tz, T^2 z\right)\right]^2 < d\left(z, Tz\right) d\left(Tz, T^2 z\right) + c.d\left(z, T^2 z\right).d\left(Tz, Tz\right).$$

Thus, $d(Tz, T^2 z) < d(z, Tz),$ which is a contradiction.

So the sequence (s_n) must converge to 0. For all n, m,

$$\begin{aligned} \left[d\left(x_n, x_m\right)\right]^2 &= \left[d\left(T^n x_0, T^m x_0\right)\right]^2 \\ &< d\left(T^{n-1}x_0, T^n x_0\right) d\left(T^{m-1}x_0, T^m x_0\right) + c.d\left(T^{n-1}x_0, T^m x_0\right) d\left(T^{m-1}x_0, T^n x_0\right) \\ &< d\left(T^{n-1}x_0, T^n x_0\right) d\left(T^{m-1}x_0, T^m x_0\right) \\ &+ c\left[d\left(T^{m-1}x_0, T^m x_0\right) + d\left(T^m x_0, T^n x_0\right)\right]\left[d\left(T^{n-1}x_0, T^n x_0\right) + d\left(T^n x_0, T^m x_0\right)\right] \\ &= s_{n-1}s_{m-1} + c\left(s_{n-1} + d\left(x_n, x_m\right)\right)\left(s_{m-1} + d\left(x_n, x_m\right)\right). \end{aligned}$$

This implies that $d(x_n, x_m) \to 0,$ as $m, n \to \infty.$

Thus, (x_n) is a Cauchy sequence. As the subsequence (x_{n_k}) of (x_n) converges to z, the limit of (x_n) must be z and $z = w$. Assume that $Tx_n \neq Tz$, we have

$$\left[d(Tx_n, Tz)\right]^2 < d(x_n, Tx_n)d(z, Tz) + cd(x_n, Tz)d(z, Tx_n).$$

Letting $n \to \infty$, we get $\left[d(z, Tz)\right]^2 \leq 0.$

This implies that $z = Tz$, i.e., z is a fixed point of T.

Next, we check the uniqueness of z. Arguing by contradiction, assume z' is another fixed point of T. Then

$$\left[d(z, z')\right]^2 = \left[d(Tz, Tz')\right]^2 < d(z, Tz)d(z'Tz') + cd(z, Tz')d(z', Tz) = c\left[d(z, z')\right]^2.$$

Since $0 \leq c < 1$, we obtain $\left[d(z, z')\right]^2 = 0$. This implies $z = z'$.

Therefore, z is the unique fixed point of T. Since we take x_0 as an arbitrary point, for every $x \in X$, the iterative sequence $(T^n x)$ converges to z, i.e., T is a Picard operator.

Example 2.1. Let $X = [1.8, \infty)$ with the usual metric $d(x, y) = |x - y|$ for all $x, y \in X$. Then, (X, d) is a compact metric space. We consider the mapping $T : X \to X$ defined by

$$Tx = \begin{cases} \dfrac{3}{4}x + \dfrac{1}{2} & \text{if } \ 1.8 \leq x < 2, \\ 2 & \text{if } \ x \geq 2. \end{cases}$$

Then T satisfies the Fisher -type contractive condition (1.1) with $c = 0.99$.

Proof.

Cases 1: If $x, y \in [1.8, \infty), x \neq y$, then $d(Tx, Ty) = 0$ and the inequality (1.1) holds.

Cases 2: If $x, y \in [1.8, 2), x \neq y$. Let $S = x + y, P = xy$. We have: $S^2 \geq 4P.$

Then $[d(Tx, Ty)]^2 = \dfrac{9}{16}(x - y)^2 = \dfrac{9}{16}(S^2 - 4P),$ and

$$\begin{aligned} d(x, Tx)d(y, Ty) + cd(x, Ty)d(y, Tx) &= \left(\frac{1}{2} - \frac{1}{4}x\right)\left(\frac{1}{2} - \frac{1}{4}y\right) + c\left|\left(x - \frac{3}{4}y - \frac{1}{2}\right)\left(y - \frac{3}{4}x - \frac{1}{2}\right)\right| \\ &= \frac{1}{16}(2 - x)(2 - y) + \frac{c}{16}\left|(4x - 3y - 2)(4y - 3x - 2)\right| \\ &= \frac{1}{16}(4 - 2S + P) + \frac{c}{16}\left|12S^2 - 49P + 2S - 4\right|. \end{aligned}$$

Thus, $d(x, Tx)d(y, Ty) + cd(x, Ty)d(y, Tx) > [d(Tx, Ty)]^2$ which is equivalent to

$(4 - 2S + P) + c\left|12S^2 - 49P + 2S - 4\right| > 9(S^2 - 4P),$ or

$$c\left|12S^2-49P+2S-4\right|>9S^2-37P+2S-4 \tag{2.2}$$

If $9S^2-37P+2S-4<0$ then (2.2) holds.

If $9S^2-37P+2S-4\geq 0$ and $c=0.99$, we have

$$0.99\left(12S^2-49P+2S-4\right)=\left(9S^2-37P+2S-4\right)+\left(2.88S^2-11.51P-0.02S+0.04\right).$$

We have

$$\begin{aligned}2.88S^2-11.51P-0.02S+0.04 &> 2.88\times 4P-11.51P-0.02S+0.04\\ &=0.01P-0.02S+0.04>0.\end{aligned}$$

Thus, $0.99\left(12S^2-49P+2S-4\right)>9S^2-37P+2S-4\geq 0.$

This implies that $0.99\left|12S^2-49P+2S-4\right|>9S^2-37P+2S-4,$ then (2.2) holds.

Cases 3: If $x\in[1.8,2), y\geq 2$,

then $[d(Tx,Ty)]^2=\left(\frac{3}{2}-\frac{3}{4}x\right)^2$ and with $c=0.99>\frac{3}{4}$, we have

$$\begin{aligned}d(x,Tx)d(y,Ty)+cd(x,Ty)d(y,Tx) &= c(2-x)\left(\frac{3}{2}-\frac{3}{4}x\right)\\ &> \frac{3}{4}(2-x)\left(\frac{3}{2}-\frac{3}{4}x\right)\\ &= \left(\frac{3}{2}-\frac{3}{4}x\right)^2=[d(Tx,Ty)]^2,\end{aligned}$$

and the inequality (1.1) holds. Hence, the inequality (1.1) holds with $c=0.99$.

Clearly, X is boundedly compact and T is a mapping satisfying (1.1). T has a fixed point and 2 is the only fixed point of T.

Theorem 2.3. *Let (X,d) be a T-orbitally compact metric space, where $T:X\to X$ is an orbitally continuous mapping such that*

$$\left[d\left(Tx,Ty\right)\right]^2<d\left(x,Tx\right)d\left(y,Ty\right)+c.d\left(x,Ty\right).d\left(y,Tx\right)$$

for all $x,y\in X$ with $x\neq y$ and $0\leq c<1$. Then, T has a unique fixed point z and for any $x\in X$, the sequence of iterates $\left(T^n x\right)$ converges to z.

Proof. Let $x_0\in X$ be arbitrary but fixed and consider the sequence $\left(x_n\right)$, where $x_n=T^n x_0$ for each $n\in\mathbb{N}$.

Since X is T-orbitally compact, the sequence $\left(x_n\right)$ has a convergent subsequence, say $\left(x_{n_k}\right)$, and let $\left(x_{n_k}\right)$ converging to z in X. By the orbital continuity of T, (Tx_{n_k}) converges to Tz.

Now, proceeding as in Theorem 2.2, we can similarly prove that the sequence $\left(d(x_n, x_{n+1})\right)$ converges to 0 and that the sequence $\left(x_n\right)$ is a Cauchy sequence and hence $x_n \to z \in X$ as $n \to \infty$. Therefore, z is the unique fixed point of T.

Theorem 2.4. *Let* (X,d) *be a complete metric space and* T *be a self-mapping on* X *such that*

1. $\left[d\left(Tx,Ty\right)\right]^2 < d\left(x,Tx\right)d\left(y,Ty\right)+c.d\left(x,Ty\right).d\left(y,Tx\right)\,(0 \le c < 1)$ *for all* x, y *in* X *with* $x \neq y$,

2. for any $x \in X$ *and for any* $\epsilon > 0$, *there exists* $\delta > 0$ *such that* $d\left(T^i x, T^j x\right) < \epsilon + \delta$ *implies* $d\left(T^{i+1}x, T^{j+1}x\right) \le \epsilon$, *for any* $i; j \in \mathbb{N} \cup \{0\}$.

Then, T *has a unique fixed point* z *and for any* $x \in X$, *the sequence of iterates* $\left(T_n x\right)$ *converges to* z.

Proof. Let $x_0 \in X$ be arbitrary but fixed and consider the sequence (x_n),

where $x_n = T^n x_0$ for each $n \in \mathbb{N}$. Let the sequence $\left(x_n\right)$ do not have two equal consecutive terms, i.e., $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$.

Then, similar to Theorem 2.1, it is not difficult to check that the sequence of real numbers $\left(s_n\right)$, where $s_n = d\left(x_n, x_{n+1}\right)$ is a decreasing sequence and also this sequence is bounded below. Thus, this sequence is convergent and let $\lim_{n\to\infty} s_n = b = \inf\{s_n, n \in \mathbb{N}\}$.

Therefore, $b \ge 0$.

Assume that $b > 0$. Then, there exists $\delta > 0$ and $n \in \mathbb{N}$ such that $s_n < b + \delta.$

This implies that $d\left(x_n, x_{n+1}\right) < b + \delta.$

Thus, by the given condition, we have $d\left(x_{n+1}, x_{n+2}\right) \le b$, i.e., $s_{n+1} \le b$, (s_n) is a decreasing sequence, so $s_{n+2} < s_{n+1} \le b$. This leads to a contradiction. Thus, we must have

$$b = \lim_{n\to\infty} d\left(x_n, x_{n+1}\right) = 0.$$

Now for any $n, m \in \mathbb{N}$, we have

$$\begin{aligned}
\left[d\left(x_n, x_m\right)\right]^2 &= \left[d\left(T^n x_0, T^m x_0\right)\right]^2 \\
&< d\left(T^{n-1}x_0, T^n x_0\right)d\left(T^{m-1}x_0, T^m x_0\right) + c.d\left(T^{n-1}x_0, T^m x_0\right)d\left(T^{m-1}x_0, T^n x_0\right) \\
&< d\left(T^{n-1}x_0, T^n x_0\right)d\left(T^{m-1}x_0, T^m x_0\right) \\
&\quad + c\left[d\left(T^{m-1}x_0, T^m x_0\right) + d\left(T^m x_0, T^n x_0\right)\right]\left[d\left(T^{n-1}x_0, T^n x_0\right) + d\left(T^n x_0, T^m x_0\right)\right] \\
&= s_{n-1}s_{m-1} + c\left(s_{n-1} + d\left(x_n, x_m\right)\right)\left(s_{m-1} + d\left(x_n, x_m\right)\right).
\end{aligned}$$

Thus, $d\left(x_n, x_m\right) \to 0$ as $m, n \to \infty.$

This implies that (x_n) is a Cauchy sequence and, we have, $\lim_{n\to\infty} x_n = z$.

Consider $Tx_n \neq Tz$, we have

$$\left[d\left(Tx_n, Tz\right)\right]^2 < d\left(x_n, Tx_n\right)d\left(z, Tz\right) + cd\left(x_n, Tz\right)d\left(z, Tx_n\right),$$

Taking the limit as $n \to \infty$, we obtain

$$\left[d\left(z, Tz\right)\right]^2 \leq 0.$$

This implies that $z = Tz$, i.e., z is a fixed point of T.

Next, we check the uniqueness of z. Arguing by contradiction, let z' be another fixed point of T, then

$$\left[d(z,z')\right]^2 = \left[d(Tz,Tz')\right]^2 < d(z,Tz)d(z'Tz') + cd(z,Tz')d(z',Tz) = c\left[d(z,z')\right]^2.$$

Since $0 \leq c < 1$, we have $\left[d(z,z')\right]^2 = 0$.

This gives $z = z'$.

References

[1] B. Fisher (1978), A fixed point theorem for compact metric spaces, *Publ. Inst.Math.* 25, 193-194.

[2] H. Garai, L.K. Dey, T. Senapati (2018), On Kannan-type contractive mappings, *Numer.Funct. Anal. Optim.* 39, 1466-1476.

[3] G.V.R. Babu (2004), Generalization of fixed point theorems relating to the diameter of orbits by using a control function, *Tamkang J. Math*., 35(2), 159-168.

[4] I. Altun, H.A. Hancer (2019), *Almost Picard operators*, AIP Conf. Proc. 2183, 060003.

[5] R.E. Edwards (1965), *Functional Analysis: Theory and Applications*, New York, NY: Holt, Rinehart and Winston.

[6] Lj. B. Ciric (1974), A generalization of Banach's contraction principle, *Proc. Amer. Math. Soc.* 45, 267-273.