GENERATION OF W-TYPE STATES IN THREE KERR-LIKE NONLINEAR OSCILLATORS SYSTEM

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Received: 03 December 2021/ Accepted: 25 March 2022/ Published: April 2022

Abstract: *In this article, we examine a model comprising three Kerr-like nonlinear oscillators with two boundaries that are pumped by external coherent fields and coupled to the center one. We demonstrate that by applying evolution operator formalism, our quantum system can be simulated and behave as a nonlinear quantum scissors, its "truncation" Hilbert space is more efficient. It will be confirmed that the model can produce bipartite and tripartite entanglement, especially W-type entangled states which is a remarkably simple quantum information problem.*

Keywords*: Entanglement, W state qubit, qutrit, negativity, quantum scissors.*

1. Introduction

The journey of Quantum Mechanics began with Planck's proposal of energy quantization and the law of radiation to explain the so-called "ultraviolet catastrophe" in the spectrum of matter. This idea was enhanced one step by Einstein when he did research on the photoelectric effect. With the time, Quantum mechanics has been improved, helping us to understand the framework of physics and its principle, it has also become a basis for advancement of many new branches of physics such as quantum information, teleportation, computation and many other fields. The essential features for such advances are known as quantum correlations and they can be affirmed to transmit, preserve and manipulate information. These led to a considerable development of particular interest in research of quantum correlations in numerous types of quantum systems. The purpose of this article is to investigate how to generate time evolution of quantum correlations in terms of the distinctive form - entanglement. Specifically, the system consists of three nonlinear Kerr-type oscillators, two boundaries are mutually coupled to center one by continuous linear interaction and excited by the external coherent fields. These oscillators can be expressed by effective Hamiltonians which are alike to those described optical Kerr systems. Quantum Kerr-type nonlinearity systems are commonly discussed in various physical applications. Such models can be put into practice in description of nanomechanical resonators and many optomechanical systems with Bose-Einstein condensate [14]. In addition, Kerr-type oscillatory models were the subject of a large number of papers related

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to the quantum chaos problems [7], [15]. The modes of Kerr-type nonlinear coupler have been proven to be auspicious devices, simple treatment for finding numerical solutions and producing entangled states and hence its quantumness.

2. The model description and discussions

The considered system comprises three nonlinear Kerr-type oscillators, they are mutually coupled to each other by linear interaction and excited by external excitation fields in two boundaries, each oscillator corresponds to a single mode of the field labeled *a, b* and *c*. It is not only the self- coupling term that exists in this system, instead of the so-called cross-Kerr coupling involving [9] which is different from many systems in previous literature [6], [7]. The total Hamiltonian describing the dynamics of our system can be defined as (assume that $\hbar = 1$).

$$\hat{H} = \hat{H}_{nl} + \hat{H}_{\text{int}} + \hat{H}_{ext}, \tag{1}$$

where

$$\hat{H}_{nl} = \frac{\chi_a}{2}\hat{a}^{\dagger 2}\hat{a}^2 + \frac{\chi_b}{2}\hat{b}^{\dagger 2}\hat{b}^2 + \frac{\chi_c}{2}\hat{c}^{\dagger 2}\hat{c}^2 + \chi_{ab}\hat{a}^{\dagger}\hat{a}\hat{b}^{\dagger}\hat{b} + \chi_{bc}\hat{b}^{\dagger}\hat{b}\hat{c}^{\dagger}\hat{c} + \chi_{ac}\hat{a}^{\dagger}\hat{a}\hat{c}^{\dagger}\hat{c}, \tag{2}$$

defines Kerr-like media (including cross-Kerr coupling); and

$$\hat{H}_{\text{int}} = \varepsilon_{ab}\hat{a}^{\dagger}\hat{b} + \varepsilon^*_{ab}\hat{a}\hat{b}^{\dagger} + \varepsilon_{bc}\hat{b}^{\dagger}\hat{c} + \varepsilon^*_{bc}\hat{b}\hat{c}^{\dagger}, \tag{3}$$

corresponds to the nonlinear interaction between coupled modes *a-b* and *b-c*;

$$\hat{H}_{ext} = \alpha\hat{a}^{\dagger} + \alpha^*\hat{a} + \gamma\hat{c}^{\dagger} + \gamma^*\hat{c} \tag{4}$$

relates to interaction with external fields.

Here χ_a, χ_b, χ_c are proportional to the third-order susceptibilities; $\chi_{ab}, \chi_{bc}, \chi_{ac}$ describe the cross - Kerr coupling processes between a - b, and b – c; whereas ε_{ab} and ε_{bc} mean the strength of the nonlinear interactions between modes a - b, and b - c. Hamiltonian of our system can be stated in terms of bosonic annihilation and creation operators, we are able to present annihilation operators as square matrices in the Hilbert space $H = H_a \otimes H_b \otimes H_c$ as follows:

$$\widehat{a} = \hat{I}_r \otimes \hat{I}_r \otimes \hat{a}, \tag{5}$$

$$\widehat{b} = \hat{I}_p \otimes \hat{b} \otimes \hat{I}_p, \tag{6}$$

$$\widehat{c} = \hat{c} \otimes \hat{I}_q \otimes \hat{I}_q, \tag{7}$$

for the mode *a*, *b* and *c,* respectively. The operators $\hat{I}_i\left(i = r, p, q\right)$ is in the form of an identity matrix with *r, p, q* dimensions for mode *a, b* and *c*.

If the dissipation processes are ignored, time-evolution of the tripartite quantum system can be expressed by wave-functions and written in the form of number photon states as:

$$\left|\psi\left(t\right)\right\rangle = \sum_{r,p,q=0}^{\infty} c_{rpq}\left|r\right\rangle_a \left|p\right\rangle_b \left|q\right\rangle_c \tag{8}$$

where the complex probability amplitudes c_{rpq} corresponds to the *r*-, *p*- and *q*- photon Fock states in mode *a, b* and *c*, correspondingly.

The evolution of our system can be regulated by a unitary operator defined from total Hamiltonian as [9]

$$\hat{U} = \exp\left(-i\hat{H}t\right). \tag{9}$$

Supposing that all interactions here are weak compared to nonlinearity constants, we can explain that the transition of the state can behave as a resonant case [14], [15]. The system's evolution is confined to the limited resonant transition states corresponding to $\{p,q,r\} = \{0,1\}$, whereas the others can be neglected. Applying the operator $\hat{U}$ on the initial state, our generated wave function can be obtained as:

$$\left|\psi(t)\right\rangle = \sum_{r,p,q=0}^{\infty} c_{rpq}\left|r\right\rangle\left|p\right\rangle\left|q\right\rangle = \hat{U}\left|\psi(0)\right\rangle. \tag{10}$$

In this work, we define the dimension of each subspace as equal to ten and assume that $\chi_a = \chi_b = \chi_c = \chi = 1$ are proportional to the third-order susceptibilities, $\chi_{ab} = \chi_{bc} = \chi_{ac} = \tilde{\chi} = 1$, and $\varepsilon_{ab} = \varepsilon_{bc} = \varepsilon$ are real number.

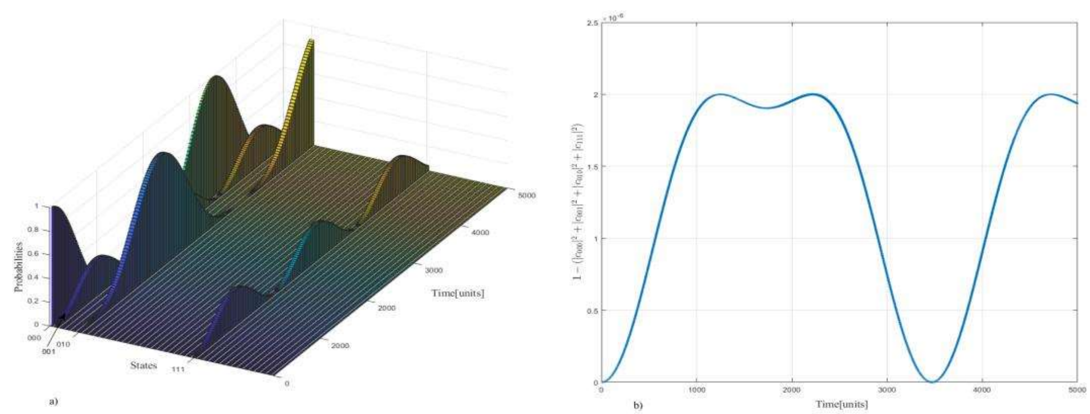

***Figure 1**. a) Time evolution of probabilities of the resonant states* $\left|0\right\rangle_a\left|0\right\rangle_b\left|0\right\rangle_c, \left|0\right\rangle_a\left|0\right\rangle_b\left|1\right\rangle_c, \left|0\right\rangle_a\left|1\right\rangle_b\left|0\right\rangle_c$, *and* $\left|1\right\rangle_a\left|0\right\rangle_b\left|0\right\rangle_c$. *b) Deviation from the unity of the sum of probabilities. The parameters are* $\chi = 1$, $\tilde{\chi} = 1$, $\alpha = \gamma = 10^{-3}$, $\varepsilon = 0,8.10^{-3}$.

As it can be seen from Fig.1a, the time evolution of probabilities of resonant states, when cross-coupling terms are taken into account. Due to effect of crossing coupling terms, the transition of the states evolved into only four states. They are $\left|0\right\rangle_a\left|0\right\rangle_b\left|0\right\rangle_c, \left|0\right\rangle_a\left|0\right\rangle_b\left|1\right\rangle_c$, $\left|0\right\rangle_a\left|1\right\rangle_b\left|0\right\rangle_c$, and $\left|1\right\rangle_a\left|0\right\rangle_b\left|0\right\rangle_c$, which are totally different from the model discussed in [6] with eight states involved in the transition. Fig.1b displays the deviation of the sum of the probabilities, the maximum deviation is $\sim 10^{-6}$. Thus, once again this confirms that the evolution of wave function is closed within four states with a high accuracy. Therefore, it can be seen that when the interacting constants are sufficiently smaller than nonlinearity coefficients, this system can be referred to as nonlinear quantum scissors. From the standpoint of quantum information theory, one can say that the discussed system can be considered as a three-qubit one, due to the fact that only two states (vacuum and one-photon state) for each of the modes are involved in the system's evolution. Thus, the wavefunction now can be written tin truncation form as:

$$\left|\psi(t)\right\rangle_{cut} = c_{000}(t)|0\rangle_a|0\rangle_b|0\rangle_c + c_{001}(t)|0\rangle_a|0\rangle_b|1\rangle_c + c_{010}(t)|0\rangle_a|1\rangle_b|0\rangle_c + c_{100}(t)|1\rangle_a|0\rangle_b|0\rangle_c$$

It is proposed that a three mode nonlinear oscillator model may generate bipartite and tripartite entangled states via optical state truncation. As measurement of bipartite entanglement, we apply the bipartite negativity which was introduced by Vidal and Werner [3], and generalized to the higher dimensions by S. Lee [13]. This quantity is an entanglement monotone and it is easy to calculate. This concept is derived from Peres–Horodecki (PPT) criterion [11], and the negativity measures negative eigenvalues of a density matrix after performing the partial transposition. If ρ_{ij} is density matrix of 2-mode system, the negativity can be obtained by the following:

$$N\left(\rho_{ij}\right) = \frac{\left\|\rho_{ij}^{T_i}\right\|_1 - 1}{2} \tag{11}$$

here $\rho_{ij} = \mathrm{Tr}_k\left(\rho_{ijk}\right); N\left(\rho_{ij}\right)$ are known as the bipartite negativity extracting from three mode matrix where the partial transpose is made for the subsystem i. For pure states, the negativity is equal to the concurrence, meanwhile this quantity gives greater values than entropy [10].

In order to distinguish among many types of tripartite entangled state which may be produced in our system, we use a Sabin's classification [2] to calculate full negativity:

$$N_{\rho_{abc}} = \left(N_{a-bc}N_{b-ac}N_{c-ab}\right)^{1/3}, \tag{12}$$

where N_{i-jk} *(i, j, k=a, b, c)* (different from (11)) can be calculated when we see *ij* as a subsystem which is equivalent to subsystem *i*.

To classify tripartite entanglement, Sabin and Garcia [2] have proposed in their report an entanglement catalog for both mix and pure states in six subtypes from fully separable state to maximal one. Type 0-0 for fully separability, type 1-1 for biseparable states, and type 2 is for fully entanglement states. For our system we are interested in type 2 with 4 subtypes as in the table below.

Table 1. *The catalog for tripartite entanglement*

reduced entanglement	type of tripartite entanglement	tripartite entangled states
$N_{ij}=N_{jk}=N_{ik}=0$	*2- 0*	GHZ-type states
$N_{ij}\neq 0; N_{jk}=N_{ik}=0$	*2- 1*	
$N_{ij}\neq 0; N_{jk}\neq 0; N_{ik}=0$	*2- 2*	star shaped-type state
$N_{ij}\neq 0; N_{jk}\neq 0; N_{ik}\neq 0$	*2- 3*	W -type class

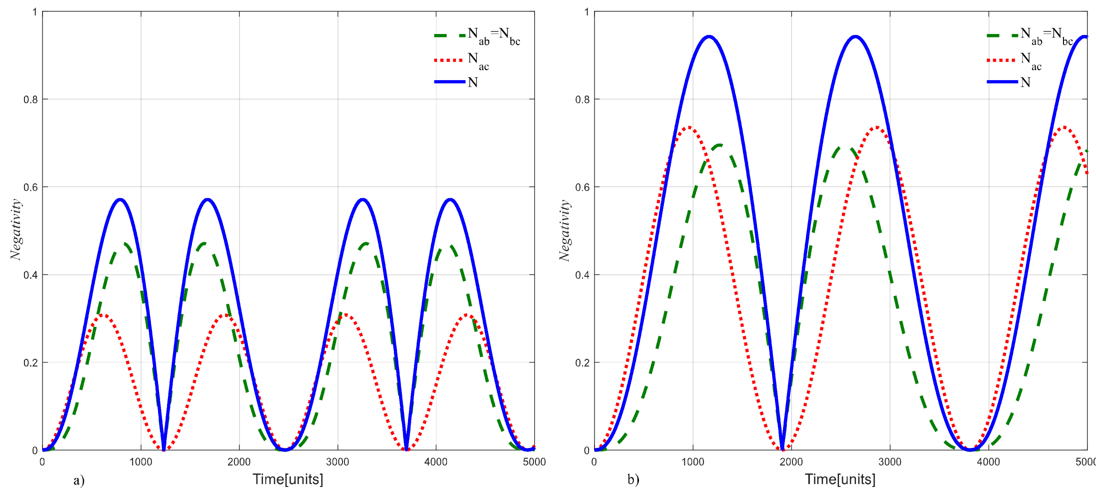

Figure 2. *Time evolution of reduced negativity and tripartite negativity for* a) $\varepsilon = 1,5.10^{-3}$ b) $\varepsilon = 0,5.10^{-3}$ and $\chi = 1$, $\tilde{\chi} = 1$, $\alpha = \gamma = 10^{-3}$.

Fig.2a and 2b show the negativity of bipartite and tripartite system with two different values of interaction strength. From these two figures, we can see that bipartite quantum entanglement is generated not only for the pair a-b, b-c (dashed-line) but also for the pair a-c (doted -line) for both values of interaction strength ε. Thus, the boundary oscillators can generate a bipartite entanglement state through interacting with the central one and depends directly on ε. Thus, by choosing the value of the magnitude of the linear interaction, we can influence the degree of the neighboring bipartite, tripartite - entanglement and time of appearance of the entanglement.

Following the time evolution, it is recognized that the maxima value of tripartite negativity is nearly unity corresponding to the smaller value of interaction (Fig.2b), this means the maximally tripartite entangled state is nearly created. For other time evolution, inseparable state is produced with smaller probabilities and associated with the W-like basic state (Fig.3) which may be expressed as [16].

$$|W_1\rangle = \frac{1}{\sqrt{3}} \; |0\rangle_a |0\rangle_b |1\rangle_c + |0\rangle_a |1\rangle_b |0\rangle_c + |1\rangle_a |0\rangle_b |0\rangle_c \;, \tag{13}$$

$$|W_2\rangle = \frac{1}{\sqrt{3}} \; |0\rangle_a |0\rangle_b |1\rangle_c - |0\rangle_a |1\rangle_b |0\rangle_c + |1\rangle_a |0\rangle_b |0\rangle_c \;.$$

When we compare these results with the ones presented in [16] for the system without crossing coupling term, in which the authors stated that the possibility of producing entangled W-states, the dynamics in our system is richer and the model can produce other types of 3-qubit entangled states with high probabilities. Though the W-type state is not a maximal entangled state, this class was confirmed having the highest robustness against the loss of one qubit [14]. The present system can be a source of generation of tripartite entangled states in the form W state, which is a remarkably simple quantum information problem to apply to quantum teleportation and quantum secure communication and so on.

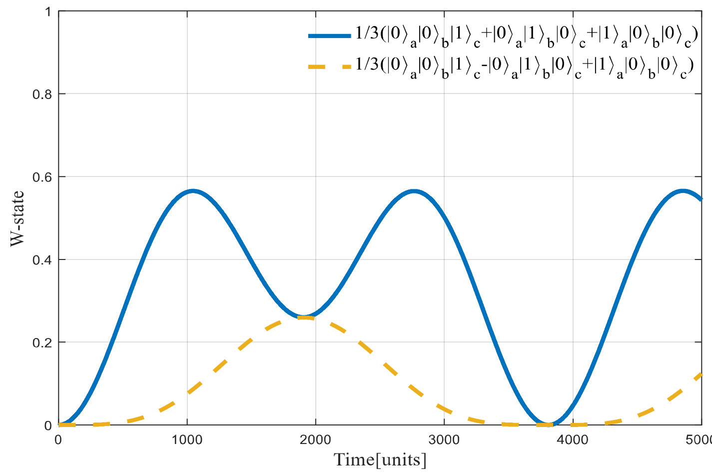

Figure 3. *Time evolution of fidelity corresponding to W-type states for* $\varepsilon = 0{,}5.10^{-3}$ *and* $\chi = 1,\ \tilde{\chi} = 1,\ \alpha = \gamma = 10^{-3}$.

3. Conclusion

In this work, we have studied the model of a chain of three nonlinear Kerr-type oscillators, with two boundaries that are coupled to the center oscillator by linear interaction and excited by external excitation fields. The system can behave as perfect three-mode nonlinear quantum scissors and can be treated as a 3-qubit system. Both bi- and tripartite entanglement have been discussed. In the case of the 2-qubit entanglement, we can detect not only entanglement between the pairs of oscillators, but also the entanglement can be generated between two oscillators even though they are not directly coupled together. For the case of 3-qubit entanglement, it is possible to obtain W-type class. Thus, the effective Hamiltonian describing our model is not only a potential source of various bi- and tri-partite entangled states, but also stimulating for the discovery of numerous types of quantum correlations as well as relations among them.

Acknowledgement: *This research was supported by Hong Duc University under grant number ĐT-2020-18.*

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