PERIODIC SOLUTION TO NONDENSELY DEFINED DIFFERENTIAL EQUATIONS WITH INFINITE DELAY

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Abstract: *In this paper, we investigate the existence and uniqueness of periodic solution to a class of nondensely defined differential equations with infinite delay of the form*

$$\begin{cases} \dfrac{du}{dt} = \left(A + B(t)\right)u(t) + g(t, u_t), \ t \geq 0 \\ u_0 = \phi \in \mathcal{B} \end{cases}$$

where $A:\mathcal{D}(A) \subset X \to X$ *is a nondensely defined linear operator on a Banach space* X *which satisfies the Hille - Yosida condition,* $(B(t))_{t\geq 0}$ *is a family of bounded linear operator,* g *is* φ *- Lipschitz function and* $\mathcal{B}$ *is appropriately phase space.*

Keywords: *Hille - Yosida condition, Periodic, Nondensely defined, Evolutionary process, Banach function space.*

1. Introduction

It is well known that a standard approach in deriving τ-periodic solutions is to define the Poincaré operator given by $P(\phi) = u_\tau(\phi)$ which maps an initial function (or value) τ-units along the unique solution $u(\phi)$ determined by the initial function (or value) ϕ. For this one, conditions are given such that some fixed point theorem can be applied to get a fixed point for the Poincaré operator, which gives rise to a periodic solution. For differential equations without delay or with finite delay in general Banach spaces, the existence of periodic solutions can be obtained by requiring that the resolvent of *A(.)* be compact, so that the abstract version of the Ascoli theorem can be used to show that the Poincaré operator is compact. Hence, the images of the Poincaré operator on bounded sets are precompact, which makes it possible to derive the periodic solutions from bounded solutions. However, this technique of showing the compactness of the Poincaré operator does not apply to differential equations with infinite delay in general Banach spaces. This means that other methods are needed to study the periodic solutions for differential equations with infinite delay in general Banach spaces such as Granas's degree theory, limiting equation technique or Kuratowski's measure of non-compactness is used to show that the Poincaré operator is condensing under some conditions, so that by Sadovskii's theorem, fixed points exist when a condensing operator maps a convex, closed, and bounded set into itself.

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Recently, for the linear equation $\dot{u} = A(t)u + f(t), t \geq 0$ the authors used a Cesàro sum to prove the existence of a periodic solution through the existence of bounded solution whose sup-norm can be controlled by the sup-norm of the input function f. Then, we use the fixed point argument to prove the existence of periodic solutions for the corresponding semi-linear problem. Especially, in [5], Huy and Dang considered the existence and uniqueness of periodic solutions to partial functional differential equations (PFDE) with infinite delay of the form

$$\dot{u} = A(t)u + g(t, u_t), \quad t \in \mathbb{R}_+$$

where for each $t \in \mathbb{R}_+, A(t)$ is a densely defined operator on a Banach space X such that the family $(A(t))_{t\geq 0}$ generates an evolution family $(U(t,s))_{t\geq s\geq 0}$ on X, and $g : \mathbb{R}_+ \times \mathcal{C}_v \to X$ is continuous and φ-Lipschitz with

$$\mathcal{C}_v := \left\{\phi : \phi \in C\left(\left(-\infty, 0\right], X\right) \text{ and } \lim_{s\to-\infty} e^{\nu s} \left\|\phi\left(s\right)\right\| = 0, \nu > 0\right\}$$

u_t is the history function defined by $u_t(\theta) = u(t+\theta)$ for $\theta \in (-\infty, 0]$. However, as indicated in [1], we sometimes need to deal with non-densely defined operators. For example, when we look at a one-dimensional heat equation with Dirichlet conditions on $[0,\pi]$ and consider $\Delta = \dfrac{\partial^2}{\partial x^2}$ in $C([0,\pi],\mathbb{R})$, in order to measure the solutions in the sup-norm, then the domain

$$\overline{\mathcal{D}(\Delta)} = \{u \in C^2([0;\pi],\mathbb{R}) : u(0) = u(\pi) = 0\} \neq C([0;\pi],\mathbb{R}).$$

More precisely, in this paper we consider a nondensely defined nonautonomous partial differential equation with infinite delay

$$\begin{cases} \dfrac{du}{dt} = (A + B(t))u(t) + g(t, u_t), & t \geq 0 \\ u_0 = \phi \in \mathcal{B} \end{cases} \tag{0.1}$$

where $A : \mathcal{D}(A) \subset X \to X$ is a nondensely defined linear operator on a Banach space X which satisfies the Hille - Yosida condition:

$\mathbf{H}_1$: there exist $M_0 \geq 1$ and $\omega_0 \in \mathbb{R}$ such that $(\omega_0, +\infty) \subset \rho(A)$ and

$$\left\|R(\xi, A)^n\right\| \leq \frac{M_0}{(\xi - \omega_0)^n}, \quad \text{for } n \in \mathbb{N} \quad \text{and} \quad \xi > \omega_0, \tag{0.2}$$

where $\rho(A)$ is the resolvent set of A and $R(\xi, A) = (\xi - A)^{-1}$; the function $g : \mathbb{R}_+ \times \mathcal{B} \to X$ is bounded continuous; for every $t \geq 0, B(t)$ is a bounded linear operator on X.

It is worth noting that when operator A is not densely defined, the linear part $A+B(t)$ does not generates a strongly continuous evolutionary process on the whole space X, so the results obtained in [5] are not guaranteed. To overcome such difficulties, we combine the methods and results in [7, 8] and appropriate choices of phase spaces to prove the existence and uniqueness of the periodic solution to (1.1) without using the uniform boundedness and smallness (in classical sense) of Lipschitz constants of the nonlinear terms.

2. Preliminaries

2.1. Notations

In this paper $\mathbb{R}, \mathbb{R}^+$ and $\mathbb{C}$ stand for the real line, its positive half line, and the complex plane. If X denotes a (complex) Banach space, then $\mathcal{L}(X)$ stands for the space of all bounded linear operators in X. The spectrum of a linear operator T in a Banach space is denoted by $\sigma(T)$, and $\rho(T) := \mathbb{C} \setminus \sigma(T)$. We denote by $C_b(\mathbb{R}^+, X)$ the space of all bounded continuous functions from $\mathbb{R}^+$ to a Banach space X and $C_b(\mathbb{R}, X)$ the space of all bounded continuous functions from $\mathbb{R}$ to a Banach space X which endowed with the supremum norm.

2.2. Mild solutions of inhomogeneous differential equations

Consider the following inhomogeneous differential equations

$$\begin{cases} \dfrac{d}{dt}x(t) = (A+B(t))x(t) + f(t), \quad \text{for } t \geq 0 \\ x(0) = x_0 \end{cases} \tag{0.3}$$

where $A: \mathcal{D}(A) \subset X \to X$ is a nondensely defined linear operator on a Banach space X which satisfies the Hille - Yosida condition. It is well known that (see [2] and the references therein) the part A_0 of A in X_0 generates a C_0-semigroup $(T_0(t))_{t\geq 0}$ on X_0 satisfying $\|T_0(t)\| \leq Me^{\omega t}, \forall t \geq 0.$ Moreover, for $\lambda \in \rho(A_0)$ the resolvent $R(\lambda, A_0)$ is the restriction of $R(\lambda, A)$ to X_0. On X_0 we introduce the norm $\|x\|_{-1} = \|R(\lambda_0, A_0)x\|$, where $\lambda_0 \in \rho(A)$ is fixed. A different choice of $\lambda_0 \in \rho(A)$ leads to an equivalent norm. The completion X_{-1} of X_0 with respect to $\|\cdot\|_{-1}$ is called the extrapolation space of X_0 with respect to A. The extrapolated semigroup $(T_{-1}(t))_{t\geq 0}$ consists of the unique continuous extensions $T_{-1}(t)$ of the operators $T_0(t)$, $t \geq 0$, to X_{-1}. The semigroup $\{T_{-1}(t)\}_{t\geq 0}$ is strongly continuous and its generator A_{-1} is the unique continuous extension of A_0 to $L(X_0, X_{-1})$. Moreover, X is continuously embedded in X_{-1} and $R(\lambda, A_{-1})$ is the unique continuous extension of $R(\lambda, A)$ to X_{-1} for $\lambda \in \rho(A)$. Finally, A_0 and A are the parts of A_{-1} in X_0 and X, respectively.

Lemma 2.1 [2]**.** For $f \in L^1_{\text{loc}}(\mathbb{R}^+, X)$ and $t \geq s \geq 0$, we have

i) $\int_s^t T_{-1}(t-\sigma) f(\sigma) d\sigma \in X_0$;

ii) $(t,s) \mapsto \int_s^t T_{-1}(t-\sigma) f(\sigma) d\sigma$ is continuous;

iii) $\left\| \int_s^t T_{-1}(t-\sigma) f(\sigma) d\sigma \right\| \leq M \int_s^t e^{\omega(t-\sigma)} \|f(\sigma)\| d\sigma$ for some constant $M \geq 1$

.

We now give the definition of a mild solution of (2.1) as follows.

Definition 2.2. Let $x_0 \in X_0$. A function $x \in C(\mathbb{R}^+, X_0)$ is called a mild solution to (2.1) if it satisfies the integral equation

$$x(t) = T_0(t-s)x(s) + \int_s^t T_{-1}(t-\sigma)\left[B(\sigma)x(\sigma) + f(\sigma)\right] d\sigma \tag{1.1}$$

for all $t \geq s \geq 0$

Now, we consider the homogeneous linear equation

$$\begin{cases} \dfrac{dx}{dt} = \left(A + B(t)\right) x(t), t \geq 0 \\ x(0) = x_0 \in X_0 \end{cases} \tag{1.2}$$

and assume that

$\mathbf{H}_2$: $t \mapsto B(t)x$ is strongly measurable for every $x \in X_0$, and there exists a function $\ell \in L^1_{loc}(\mathbb{R}^+)$ such that $\|B(\cdot)\| \leq \ell(\cdot).$

$\mathbf{H}_3$: The operator $B(\cdot)$ is τ-periodic.

Proposition 2.3 ([2]). Let $\mathbf{H}_1 - \mathbf{H}_3$ be satisfied. Then, there exists a unique τ-periodic strongly continuous evolutionary process $\mathcal{U}_B(t,s)_{t\geq s\geq 0}$ that satisfies

1. $\mathcal{U}_B(t,s)\in\mathcal{L}(X_0)$ for all $t\geq s\geq 0$;
1. $\mathcal{U}_B(t,t)=I$, for every $t\in\mathbb{R}$;
2. $\mathcal{U}_B(t,s)\mathcal{U}_B(s,r)=\mathcal{U}_B(t,r)$, for all $t\geq s\geq r$;
3. $\mathcal{U}_B(t+\tau,s+\tau)=\mathcal{U}_B(t,s)$ for all $t\geq s\geq 0$;
4. The function $(t,s,x)\mapsto \mathcal{U}_B(t,s)x$ is continuous in t,s,x;
5. There are positive constants R,δ such that
$$\|\mathcal{U}_B(t,s)\|\leq Re^{\delta(t-s)}, \text{ for all } t\geq s\geq 0.$$
6. Furthermore,

$$\mathcal{U}_B(t,s)x=T_0(t-s)x+\int_s^t T_{-1}(t-\sigma)B(\sigma)\,\mathcal{U}_B(\sigma,s)xd\sigma,\ t\geq s\geq 0, x\in X_0. \tag{2.4}$$

7. i.e., $t\mapsto \mathcal{U}_B(t,0)x_0$ is the unique solution of (2.3).

Remark 2.4. The representation of mild solution of (2.1) as in (2.2) is not convenient to investigate the periodicity. To overcome this difficulty, we rewrite the mild solution in terms of periodic strongly continuous evolutionary process $\mathcal{U}_B(t,s)_{t\geq s\geq 0}$ as follows.

Theorem 2.5 ([2]). Let $f\in L^1_{loc}$ and $x_0\in X_0$. Then there is a unique mild solution $x(\cdot)\in C(\mathbb{R}^+, X_0)$ of Equation (2.1) which satisfies the integral equation

$$x(t)=\mathcal{U}_B(t,s)x(s)+\lim_{\lambda\to\infty}\int_s^t \mathcal{U}_B(t,\sigma)\lambda R(\lambda,A)f(\sigma)d\sigma \quad \text{for } t\geq s\geq 0.$$

Moreover, $\lim\limits_{\lambda\to\infty}\int_s^t \mathcal{U}_B(t,\tau)\lambda R(\lambda,A)f(\sigma)d\sigma\in X_0$ exists uniformly for $t\geq s$ in compact sets in $\mathbb{R}$.

2.3. Phase space for infinite delay evolution equations

In this paper, we will use an axiomatic definition of the phase space $\mathcal{B}$ introduced by Hale and Kato in [3] and follow the terminology used in [4]. Thus, $\mathcal{B}$ is a linear space of functions mapping $(-\infty;0]$ into X endowed with a norm $\|\cdot\|_{\mathcal{B}}$.

We assume that $\mathcal{B}$ satisfies the following axioms: $(\mathbf{A})$: If $x\colon -\infty,\sigma+a \mapsto X, a>0$ is continuous on $\sigma;\sigma+a$ and $x_\sigma \in \mathcal{B}$, then for every $t\in \sigma;\sigma+a$ the following conditions hold:

1. x_t is in $\mathcal{B}$.

[1] $\|x(t)\| \leq H\|x_t\|_{\mathcal{B}}$.

[2] $\|x_t\|_{\mathcal{B}} \leq K(t-\sigma)\sup \|x(s)\|:\sigma\leq s\leq t + M(t-\sigma)\|x_\sigma\|_{\mathcal{B}}$

Here $H\geq 0$ is a constant, $K,M\colon 0;+\infty \to 0;+\infty$, $K(\cdot)$ is continuous and $M(\cdot)$ is locally bounded, and $H,K(\cdot),M(\cdot)$ are independent of $x(\cdot)$.

$(\mathbf{A}_1)$: For the function $x(\cdot)$ in $(\mathbf{A})$, the function $t\mapsto x_t$ is continuous from $\sigma;\sigma+a$ into $\mathcal{B}$.

$(\mathbf{B})$: The space $\mathcal{B}$ is complete.

We make the following assumption

$\mathbf{H}_4$: The exists positive constants $\hbar,\kappa$ such that

$$\hbar\|x(t)\| \leq \|x_t\|_{\mathcal{B}} \leq \kappa \sup_{t\in\mathbb{R}^+}\|x(t)\| \tag{2.5}$$

Example 2.6. If $\mathcal{B}$ is a uniformly fading memory space, then (2.5) is fulfilled.

We finish this section by recall some notions on Banach function spaces and its admissibility. We denote

$$\mathbf{M} = \mathbf{M}\ \mathbb{R}^+ := \left\{ f\in L_{1,\mathrm{loc}}\ \mathbb{R}^+ \ \middle|\ \sup_{t\geq 0}\int_t^{t+1} |f(\sigma)|\,d\sigma < \infty \right\}$$

endowed with the norm

$$\|f\|_{\mathbf{M}} := \sup_{t\geq 0}\int_t^{t+1}\|f(\sigma)\|\,d\sigma.$$

Clearly, $\mathbf{M}$ is a Banach space and it is an admissible Banach function space in the sense of [6, Definition 1.2]

For a given Banach space X, we define the space $\mathfrak{M}$ of X-valued functions related to $\mathbf{M}$ by

$$\mathfrak{M} := f:\mathbb{R}^+ \mapsto X \,\big|\, \|f(\cdot)\| \in \mathbf{M}$$

endowed with the norm $\|f\|_{\mathfrak{M}} := \big\|\|f(\cdot)\|\big\|_{\mathbf{M}}$. Clearly, $\mathfrak{M}$ is a Banach space. Moreover, we consider the following subset of $\mathbf{M}$ consisting of 1-periodic functions denoted by

$$\mathrm{P} := \{ f \in \mathbf{M} \mid f \text{ is } \tau\text{-periodic} \}$$

We put

$$\mathcal{B}_A := \{ \phi \in \mathcal{B} : \phi(0) \in X_0 \}.$$

and for $x \in X, \tilde{\phi} \in \mathcal{B}$, $\tilde{\omega} \in C_b(\mathbb{R}, X)$ we define the following subsets,

$$B_\epsilon(x) := \{ z \in X_0 : \| x - z \| \leq \epsilon \};$$

$$\mathbb{B}_\epsilon(\tilde{\phi}) := \{ \phi \in \mathcal{B}_A : \| \tilde{\phi} - \phi \| \leq \epsilon \};$$

$$\mathcal{B}_\epsilon(\tilde{\omega}) := \{ \omega \in C_b(\mathbb{R}, X) : \omega_0 \in \mathcal{B}_A \text{ and } \| \tilde{\omega} - \omega \|_{C_b} \leq \epsilon \}.$$

Then, we give the following definition.

Definition 2.7. Let $\varphi \in \mathbf{M}$ be a positive function. A function $F: [0;+\infty) \times \mathrm{B}_\alpha(0) \to X$ is said to belong to the class (L,φ,α), for some constants L,α if F satisfies:

a) $\|F(t,0)\| \leq L\varphi(t)$ for a.e $t \in \mathbb{R}^+$,

b) $\|F(t,\chi_1) - F(t,\chi_2)\| \leq \varphi(t)\|\chi_1 - \chi_2\|_{\mathcal{B}}$, for all $\chi_1, \chi_2 \in \mathcal{B}_\alpha(0)$, a.e in $t \in \mathbb{R}^+$.

3. Main results

3.1. Periodic solution for the semilinear equation.

Now we are in situation to investigate the periodicity of solutions to the following nondensely defined nonautonomous partial differential equation

$$\begin{cases} \dfrac{du}{dt} = (A + B(t))u(t) + g(t,u_t), \ t \geq 0, \\ u_0 = \phi \in \mathcal{B}_A. \end{cases} \tag{3.1}$$

We give the definition of a mild solution to Equation (3.1) as follows.

Definition 3.1. A function $u: \mathbb{R} \to X$ is a mild solution of equation (3.1), if $u_0 = \phi \in \mathcal{B}_A$ and $u(\cdot)$ is continuous on $[0;+\infty)$ satisfying that

$$u(t) = T_0(t)\phi(0) + \int_0^t T_{-1}(t-\sigma)\big(B(\sigma)u(\sigma) + g(\sigma,u_\sigma)\big)d\sigma \quad \text{for} \quad t \geq 0. \tag{3.2}$$

We assume that

$(\mathbf{H}_5)$: g belongs to the class (L,φ,α) for $h,\alpha > 0$ and $0 < \varphi \in \mathbf{M}$.

Assume that $(\mathbf{H}_1)-(\mathbf{H}_5)$ hold. Let $\phi \in \mathcal{B}_A$. Then, for $\|\varphi\|_{\mathbf{M}}$ small enough, Equation (3.1) has a unique mild solution $u \in \mathcal{B}_\alpha(0)$, given by

$$\begin{cases} u(t) = \mathcal{U}_B(t,0)\phi(0) + \lim\limits_{\lambda\to\infty}\int_0^t \mathcal{U}_B(t,\sigma)\lambda R(\lambda,A)g(\sigma,u_\sigma)d\sigma, & \text{for } t \ge 0 \\ u_0 = \phi. \end{cases} \tag{3.3}$$

Moreover, the limit

$$\lim_{\lambda\to\infty}\int_0^t \mathcal{U}_B(t,\sigma)\lambda R(\lambda,A)F(\sigma,u_\sigma)d\sigma \in X_0,$$

exists uniformly on compact sets on $\mathbb{R}_+$ and mild solution depends continuously on the initial data ϕ.

Proof. Let $t_1 > 0$ and define the closed subset

$$\mathcal{B}_{t_1,\alpha} := \left\{ w \in C_b((-\infty,t_1],X):\ w_0 = \phi,\ w(0) \in X_0 \text{ and } \sup_{-\infty<t\le t_1}\|w(t)\| \le \alpha \right\}$$

endowed with the norm $\|w\|_{\mathcal{B}_{t_1,\alpha}} = \sup\limits_{-\infty<t\le t_1}\|w(t)\|.$ Now, for $w \in \mathcal{B}_{t_1,\alpha}$ we define

$$(\mathcal{F}w)(t) := \mathcal{U}_B(t,0)w(0) + \lim_{\lambda\to\infty}\int_0^t \mathcal{U}_B(t,\sigma)\lambda R(\lambda,A)g(\sigma,w_\sigma)d\sigma.$$

Then, $\mathcal{F}w \in C_b((-\infty;t_1],X)$ and

$$\begin{aligned} \|(\mathcal{F}w)(t)\| &\le \|\mathcal{U}_B(t,0)w(0)\| + \left\|\lim_{\lambda\to\infty}\int_0^t \mathcal{U}_B(t,\sigma)\lambda R(\lambda,A)g(\sigma,w_\sigma)d\sigma\right\| \\ &\le Re^{\delta t_1}\|w(0)\| + R(L+\kappa\alpha)e^{\delta t_1}[t_1+1]\|\varphi\|_{\mathbf{M}}. \end{aligned}$$

Thus, choosing t_1 and $\|\varphi\|_{\mathbf{M}}$ small enough such that $\sup\limits_{-\infty<t\le t_1}\|(\mathcal{F}w)(t)\| \le \alpha$, we obtain that $(\mathcal{F}w) \in \mathcal{B}_{t_1,\alpha}$. On the other hand, let $v,w \in \mathcal{B}_{t_1,\alpha}$, then

$$\begin{aligned} \|(\mathcal{F}v)(t)-(\mathcal{F}w)(t)\| &\le \left\|\lim_{\lambda\to\infty}\int_0^t \mathcal{U}_B(t,\sigma)\lambda R(\lambda,A)\big(g(\sigma,v_\sigma)-g(\sigma,w_\sigma)\big)d\sigma\right\| \\ &\le Re^{\delta t_1}\|v_\sigma - w_\sigma\|_{\mathcal{B}}[t_1+1]\|\varphi\|_{\mathbf{M}} \\ &\le Re^{\delta t_1}\kappa[t_1+1]\|\varphi\|_{\mathbf{M}}\|v-w\|_{\mathcal{B}_{t_1,\alpha}}. \end{aligned}$$

Hence, it follows that,

$$\|\mathcal{F}v-\mathcal{F}w\|_{\mathcal{B}_{t_1,\alpha}} \le Re^{\delta t_1}\kappa[t_1+1]\|\varphi\|_{\mathbf{M}}\|v-w\|_{\mathcal{B}_{t_1,\alpha}}.$$

Choosing t_1 and $\|\varphi\|_{\mathbf{M}}$ small enough, we obtain that F is a contraction. Thus, there is a unique $v \in \mathcal{B}_{t_1,\alpha}$ such that $\mathcal{F}v = v.$ Remark that for $t \in [0,t_1],\ v(t) \in C_b([0,t_1], X_0)$ by construction. Particularly, $v(t_1) \in X_0$. Then, proceeding inductively on $[t_n, t_{n+1}]$, for $n \in \mathbb{N}^*$ and repeating the same work, with taking $w_0 = v_{t_n}$, we obtain that there is a unique $v \in \mathcal{B}_\alpha(0)$ satisfying $\mathcal{F}v = v$. Now, we will prove that such solution v is the unique mild solution of (3.1) in $\mathcal{B}_\alpha(0)$. In fact, setting

$$z_\lambda(t) := \int_0^t \mathcal{U}_B(t,\sigma)\lambda R(\lambda, A) g(\sigma, v_\sigma) d\sigma$$

We have in view of (2.4), that

$$z_\lambda(t) = \lambda R(\lambda, A_0)\int_0^t T_{-1}(t-\sigma) g(\sigma, v_\sigma) d\sigma + \int_0^t T_{-1}(t-\sigma) B(\sigma) z_\lambda(\sigma) d\sigma. \tag{3.4}$$

As $v \in \mathcal{B}_\alpha(0)$, then, by $(\mathbf{H}_5)$,

$$\|g(\sigma, v_\sigma)\| \le (L + \kappa\alpha)\varphi(\sigma).$$

Since $\varphi \in L^1_{loc}(\mathbb{R}^+)$, then $\sigma \mapsto g(\sigma, v_\sigma) \in L^1_{loc}(\mathbb{R}^+)$ Putting

$$w(t) = \int_0^t T_{-1}(t-\sigma) g(\sigma, v_\sigma) d\sigma.$$

We have by Lemma (2.1),

$$\|z_\mu(t) - z_\nu(t)\| \le \|(\mu R(\mu, A_0) - \nu R(\nu, A_0)) w(t)\| + \tag{3.5}$$
$$+ M\int_0^t e^{\omega(t-\sigma)} \ell(\sigma) \|z_\mu(\sigma) - z_\nu(\sigma)\| d\sigma$$

According to Lemma (2.1), $w(t)$ is continuous into X_0. Consequently,

$$\lim_{\mu,\nu \to \infty} \|(\mu R(\mu, A_0) - \nu R(\nu, A_0)) w(t)\| = 0$$

uniformly in compact intervals, for $t \ge 0$.

From (1.2), we deduce that for $\varepsilon > 0$ and $I \subseteq \mathbb{R}$ a compact interval, there is a constant N depending on the length of I such that

$$\|z_\mu(t) - z_\nu(t)\| \le \varepsilon + N\int_0^t \ell(\sigma) \|z_\mu(\sigma) - z_\nu(\sigma)\| d\sigma$$

for $t \geq 0$ in I and $\mu, \nu > w$ large enough.

An application of Gronwall's inequality gives

$$\left\| z_\mu(t) - z_\nu(t) \right\| \leq \varepsilon e^{N\int_0^t \ell(\sigma)d\sigma},$$

for $t \geq 0$ in I and $\mu, \nu > w$ large enough. Thus, $z(t) = \lim\limits_{\lambda\to\infty} z_\lambda(t)$ exists uniformly for $t \geq 0$ in compact intervals. Since $(\mathbf{H}_1)$, it yields from the definition of z_λ that

$$\sup\{z_\lambda(t) : \lambda > w, t > 0 \text{ in } I\} < \infty.$$

Applying the Lebesgue convergence theorem to (3.4), we obtain

$$z(t) = \int_0^t T_{-1}(t-\sigma)B(\sigma)z(\sigma)d\sigma + \int_0^t T_{-1}(t-\sigma)g(\sigma, v_\sigma)d\sigma, \quad t \geq 0 \tag{3.6}$$

Hence, using (2.4) and (3.6), we have

$$\begin{aligned}
v(t) &= \mathcal{U}_B(t,0)\phi(0) + \lim_{\lambda\to\infty}\int_0^t \mathcal{U}_B(t,\sigma)\lambda R(\lambda, A)g(\sigma, v_\sigma)d\sigma \\
&= \mathcal{U}_B(t,0)\phi(0) + z(t) \\
&= \mathcal{U}_B(t,0)\phi(0) + \int_0^t T_{-1}(t-\sigma)B(\sigma)z(\sigma)d\sigma + \int_0^t T_{-1}(t-\sigma)g(\sigma, v_\sigma)d\sigma \\
&= T_0(t)\phi(0) + \int_0^t T_{-1}(t-\sigma)B(\sigma)\left(\mathcal{U}_B(\sigma,0)\phi(0) + z(\sigma)\right)d\sigma + \int_0^t T_{-1}(t-\sigma)g(\sigma, v_\sigma)d\sigma \\
&= T_0(t)\phi(0) + \int_0^t T_{-1}(t-\sigma)\left(B(\sigma)v(\sigma) + g(\sigma, v_\sigma)\right)d\sigma, \quad t \geq 0,
\end{aligned}$$

which means that v is a mild solution of (3.1).

Now, let u and v be two mild solutions of (3.1) in $\mathcal{B}_\alpha(0)$, such that $u_0 = \phi_1$ and $v_0 = \phi_2$. Then, for $0 \leq t \leq T$, we have

$$\begin{aligned}
\|u(t)-v(t)\| &= \left\| T_0(t)\left(\phi_1(0)-\phi_2(0)\right) + \int_0^t T_{-1}(t-\sigma)\left(B(\sigma)(u(\sigma)-v(\sigma)) + g\left(\sigma, u_\sigma\right) - g\left(\sigma, u_\sigma\right)\right)d\sigma \right\| \\
&\leq \left\|T_0(t)\left(\phi_1(0)-\phi_2(0)\right)\right\| + \left\|\int_0^t T_{-1}(t-\sigma)B(\sigma)(u(\sigma)-v(\sigma))d\sigma\right\| + \left\|\int_0^t T_{-1}(t-\sigma)\left(g(\sigma,u_\sigma)-g(\sigma,u_\sigma)\right)d\sigma\right\| \\
&\leq \left\|T_0(t)\left(\phi_1(0)-\phi_2(0)\right)\right\| + \int_0^t \left\|T_{-1}(t-\sigma)B(\sigma)\right\| \|u(\sigma)-v(\sigma)\| d\sigma + M\int_0^t e^{\omega(t-\sigma)}\varphi(\sigma)\|u_\sigma - v_\sigma\|_{\mathcal{B}} d\sigma \\
&\leq \frac{Me^{\omega T}}{\hbar}\|\phi_1-\phi_2\|_{\mathcal{B}} + \int_0^t \max\left(\frac{\|T_{-1}(t-\sigma)B(\sigma)\|}{\hbar}, Me^{\omega(t-\sigma)}\varphi(\sigma)\right)\|u_\sigma - v_\sigma\|_{\mathcal{B}}\, d\sigma.
\end{aligned} \tag{3.7}$$

Now, from (3.7) we obtain

$$\|u_t - v_t\|_{\mathcal{B}} \le \max\left(1, \frac{Me^{\omega T}}{\hbar}\right)\|\phi_1 - \phi_2\|_{\mathcal{B}}$$

$$+\int_0^T \max\left(\frac{\|T_{-1}(t-\sigma)B(\sigma)\|}{\hbar}, Me^{w(t-\sigma)}\varphi(\sigma)\right)\|u_\sigma - v_\sigma\|_{\mathcal{B}}\, d\sigma, \text{ for } 0 \le t \le T.$$

Using the Gronwall's inequality, we have

$$\|u_t(,\phi_1) - v_t(.,\phi_2)\|_{\mathcal{B}} \le \max\left(1, \frac{Me^{\omega T}}{\hbar}\right)e^{CT}\|\phi_1 - \phi_2\|_{\mathcal{B}}, \quad \text{for} \quad 0 \le t \le T,$$

for some $C > 0$, which implies the uniqueness of v and the continuity of the map $\phi \mapsto u_t(.,\phi)$ uniformly for $t \in [0,T]$. Proceeding inductively, we get the uniqueness of v and the continuity of the map $\phi \mapsto u_t(.,\phi)$ uniformly for $t \in [0,+\infty)$.

$(\mathbf{H}_6)$: The Banach space $X_0 = Y'$ for a separable Banach space Y, and Y which is a subspace of Y'' is invariant under the operator $\mathcal{U}_B'(\tau,0)$, the dual of $\mathcal{U}_B(\tau,0)$.

Theorem 3.3 ([7]). *Assume that* $(\mathbf{H}_1)-(\mathbf{H}_3)$ *and* $(\mathbf{H}_6)$ *hold and* $f \in \mathbf{P}$. *If Equation (2.1) has a bounded mild solution* $u(\cdot)$ *on* $\mathbb{R}_+$ *such that*

$$\|u\|_{C_b(\mathbb{R}_+,X_0)} \le C\|f\|_{\mathbf{M}} \tag{3.8}$$

for some constant $C > 0$, *then it has a* $\tau-$*periodic mild solution* $\tilde{u}(\cdot)$ *satisfying*

$$\|\tilde{u}\|_{C_b(\mathbb{R}_+,X_0)} \le R\left(C + [\tau] + 1\right)e^{\delta\tau}\|f\|_{\mathbf{M}} \tag{3.9}$$

where $[\tau]$ *is the floor part of* τ.

Further, if the evolution family $(\mathcal{U}_B(t,s))_{t\ge s\ge 0}$ *satisfies*

$$\lim_{t\to\infty}\|\mathcal{U}_B(t,0)x\| = 0 \text{ for } x \in X_0 \text{ such that } \mathcal{U}_B(t,0) \text{ is bounded on } \mathbb{R}_+, \tag{3.10}$$

then the τ*-periodic mild solution is unique.*

$(\mathbf{H}_7)$: The function $g(.,\phi)$ is τ-periodic, for each $\phi \in \mathcal{B}$.

Theorem 3.4. *Assume that* $(\mathbf{H}_1)-(\mathbf{H}_7)$ *hold. If for every* $f \in \mathbf{P}$, *there exists a bounded mild solution* u *of* (2.1) *such that*

$$\|u\|_{C_b(\mathbb{R}_+,X_0)} \le C\|f\|_{\mathbf{M}}$$

and the evolution family $(\mathcal{U}_B(t,s))_{t\ge s\ge 0}$ *satisfies*

$\lim_{t\to\infty}\|\mathcal{U}_B(t,0)\mu\|=0$ for all $\mu\in X_0$ such that $\mathcal{U}_B(t,0)\mu$ is bounded on $\mathbb{R}^+$.

Then, if $\|\varphi\|_{\mathbf{M}}$ *is sufficiently small, Equation (3.1) has a unique* $\tau-$ *periodic solution in* $\mathcal{B}_\alpha(0)$.

Proof. Firstly, we define closed subset

$$\mathcal{B}_\alpha^\tau := \left\{ w\in\mathcal{B}_\alpha(0) : w \text{ is } \tau-\text{periodic} \right\},$$

endowed with the norm $\|w\|_{C_b} = \sup_{t\in\mathbb{R}}\|w(t)\|$.

Then, for $w\in\mathcal{B}_\alpha^\tau$, let $u(\cdot)$ be defined by

$$u(t)=\mathcal{U}_B(t,0)u(0)+\lim_{\lambda\to\infty}\int_0^t \mathcal{U}_B(t,\sigma)\lambda R(\lambda,A)g(\sigma,w_\sigma)d\sigma \tag{3.11}$$

for $u(0)\in X_0$. Note that for $w\in\mathcal{B}_\alpha^\tau$, since w is $\tau-$periodic, use $(\mathbf{H}_4)$ we have that $w_t\in\mathcal{B}_A$ and

$$\| w_t \|_{\mathcal{B}} \le \kappa \| w(\cdot) \|_{C_b(\mathbb{R},X)} \le \kappa\alpha.$$

If we put $f(t):=g(t,w_t)$, then

$$\|g(s,w_s)\| = \sup_{t\ge 0}\int_t^{t+1}\|g(s,w_s)\|ds \le \sup_{t\ge 0}\int_t^{t+1}\left(\|g(s,0)\|+\|g(s,w_s)-g(s,0)\|\right)ds$$

$$\le \left(L+\kappa\alpha\right)\sup_{s\ge 0}\int_t^{t+1}\|\varphi(s)\|\,ds \le \left(L+\kappa\alpha\right)\|\varphi\|_{\mathbf{M}}.$$

It follows that $f\in\mathbf{P}.$ Moreover, f is $\tau-$periodic by hypothesis $(\mathbf{H}_7)$ and the fact that w is $\tau-$periodic. Now, an application of Theorem 3.3 guarantees that there exists a unique $\tau-$periodic solution u for (3.11) satisfying

$$\|u\|_{C_b(\mathbb{R},X_0)} \le R\left(C+[\tau]+1\right)e^{\delta\tau}\left(L+\kappa\alpha\right)\|\varphi\|_{\mathbf{M}}.$$

Now, for $w\in\mathcal{B}_\alpha^\tau$ we define

$$\Psi(w)(t)=\begin{cases} u(t), & \text{for all } t\ge 0\\ \tilde{u}(t), & \text{for all } t<0\end{cases}$$

where u is the unique $\tau-$periodic solution for (3.11) and $\tilde{u}(\cdot)$ is the unique $\tau-$periodic extension of $u(\cdot)$ on $\mathbb{R}^-$. Since $\tilde{u}$ is the $\tau-$periodic extension of u on $\mathbb{R}^-$,

$$\| \Psi(w) \|_{C_b(\mathbb{R},X_0)} = \| u \|_{C_b(\mathbb{R}_+,X_0)} \le R(C+[\tau]+1)e^{\delta\tau}\left(L+\kappa\alpha\right)\| \varphi \|_{\mathbf{M}}.$$

Therefore, if $\|\varphi\|_{\mathbf{M}}$ is small enough, then Ψ acts from $\mathcal{B}_\alpha^\tau$ into itself.

Now, by (3.11) we have the following representation of Ψ

$$\Psi(w)(t) = \begin{cases} \mathcal{U}_B(t,0)u(0) + \lim\limits_{\lambda\to\infty}\int\limits_0^t \mathcal{U}_B(t,\sigma)\lambda R(\lambda, A)g(\sigma, w_\sigma)d\sigma & \text{for } t \geq 0, \\ \tilde{u}(t) & \text{for } t < 0 \end{cases}$$

where, as above, the function $\tilde{u}(t)$ is the $\tau-$periodic extension to interval $(-\infty, 0)$ of the periodic function

$$u(t) = \mathcal{U}_B(t,0)u(0) + \lim_{\lambda\to\infty}\int\limits_0^t \mathcal{U}_B(t,\sigma)\lambda R(\lambda, A)g(\sigma, w_\sigma)d\sigma \text{ for } t \geq 0.$$

Furthermore, for $w_1, w_2 \in \mathcal{B}_\alpha^\tau$. Then, $u = \Psi(w_1) - \Psi(w_2) = u_1 - u_2$ is the unique $\tau-$periodic solution to the equation

$$\begin{cases} u(t) = \mathcal{U}_B(t,0)u(0) + \lim\limits_{\lambda\to\infty}\int\limits_0^t \mathcal{U}_B(t,\sigma)\lambda R(\lambda, A)\left(g(\sigma, w_{1\sigma}) - g(\sigma, w_{2\sigma})\right)d\sigma, \text{ for } t \geq 0 \\ u(t) = \tilde{u}(t) = \tilde{u}_1(t) - \tilde{u}_2(t), \quad \text{for} \quad t < 0 \end{cases}$$

Since $u(t), t \geq 0$, is $\tau-$periodic, and for $t < 0$ the function $\tilde{u}(t)$ is an $\tau-$periodic extension of u to interval $(-\infty, 0)$, we have that

$$\begin{aligned} \|\Phi(w_1) - \Phi(w_2)\|_{C_b(\mathbb{R},X)} &= \sup_{t\in\mathbb{R}}\|u(t)\| = \sup_{t\geq 0}\|u(t)\| \\ &\leq R(C + [\tau] + 1)e^{\delta\tau}\sup_{t\geq 0}\int_t^{t+1}\|g(\sigma, w_{1\sigma}) - g(\sigma, w_{2\sigma})\|d\sigma \\ &\leq R(C + [\tau] + 1)e^{\delta\tau}\|\varphi\|_{\mathbf{M}}\|w_{1\sigma} - w_{2\sigma}\|_{\mathcal{B}}. \end{aligned}$$

Hence, since w_1 and w_2 are $\tau-$periodic functions, from $(\mathbf{H}_4)$ we obtain

$$\|w_{1\sigma} - w_{2\sigma}\|_{\mathcal{B}} \leq \kappa\|w_1 - w_2\|_{C_b(\mathbb{R},X)} \quad \text{for all } t \geq 0.$$

Then,

$$\|\Psi(w_1) - \Psi(w_2)\|_{C_b(\mathbb{R},X)} \leq R\left(C + [\tau] + 1\right)e^{\delta\tau}\kappa\|\varphi\|_{\mathbf{M}}\|w_1 - w_2\|_{C_b(\mathbb{R},X)}.$$

Thus, if $\|\varphi\|_{\mathbf{M}}$ is small enough, then $\Psi\colon \mathcal{B}_\alpha^\tau \to \mathcal{B}_\alpha^\tau$ is a contraction and an application of the Banach fixed point theorem yields that there exists a unique bounded periodic function u in $\mathcal{B}_\alpha^\tau$ such that $\Psi(\hat{u}) = \hat{u}$. Consequently, from the definition of Ψ, it follows that $\hat{u}$ is the solution of (3.1). This gives the result.

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