# STABILIZATION OF TWO-DIMENSIONAL SINGULAR ROESSER SYSTEMS

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**Abstract:** *This paper is concerned with the stabilization problem via state-feedback control for a class of two-dimensional (2-D) singular Roesser systems. Based on a 2-D Lyapunov function scheme, and by utilizing zero-type free matrix equations, sufficient conditions in the form of linear matrix inequalities (LMIs) are first derived to guarantee the admissibility (causality and asymptotic stability) of the closed-loop systems. Then, a stabilizing state-feedback controller (SFC) can be implemented using tractable LMIs conditions.*

**Keywords:** *2-D systems, Roesser model, singularity, linear matrix inequalities.*

## 1. Introduction

Two-dimensional (2-D) systems are widely used to describe dynamics of various practical models in control engineering. Typical applications of 2-D systems theory can be found in, for example, image processing, geographical data processing, electricity transmission, gas absorption, water stream heating or air drying [5-8]. Thus, the study of 2-D systems, both in theory and application design, has attracted considerable attention from researchers during the past few decades. We refer the reader to [10, 12, 13] just for a few references. In particular, there have been a few results concerning stability and stabilization of 2-D systems.

For example, in [11], the stability of 2-D Roesser systems with time-varying delays have been studied. In [12], the authors investigated the problem of $H_\infty$ stabilization of 2-D switch systems. The authors of [22] addressed the energy-to-peak stability of 2-D time-delay Roesser systems with multiplicative stochastic noises.

On the other hand, singular systems (also known as algebraic or descriptor systems) are widely used to describe dynamics of various practical phenomena such as electrical circuit networks, power systems, multibody mechanics, aerospace engineering, and chemical and physical processes [1-4]. In the past few decades, considerable effort from researchers has been devoted to the study of stability analysis and control of singular systems and many results have been reported in the literature. To mention a few, we refer the reader to [14, 15, 17] for the problem of stability analysis and [16, 18, 19] for some other control issues related to singular delayed systems. However, to the best of the author knowledge, the problemof stability for 2-D systems has not been fully investigated to date. This motivates the present study.

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*Notation:* $\mathbb{R}^{n\times m}$ denotes the set of $n\times m$ real matrices and $diag(A,B) \overset{\Delta}{=} \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$ for two matrices $A, B$ of appropriate dimensions. $Sym(A) \overset{\Delta}{=} A + A^T$ for $A \in \mathbb{R}^{n\times m}$. A matrix $M \in \mathbb{R}^{n\times m}$ is semi-positive definite, $M \geq 0$ if $x^T Mx \geq 0, \quad \forall x \in \mathbb{R}^{n\times n}$; $M$ is positive definite, $M > 0$, if $x^T Mx > 0, \quad \forall x \in \mathbb{R}^{n\times n}, x \neq 0.$

## 2. Preliminaries

Consider a class of 2-D singular systems described by the following Roesser model

$$E\begin{bmatrix} x^h(i+1,j) \\ x^v(i,j+1) \end{bmatrix} = A\begin{bmatrix} x^h(i,j) \\ x^v(i,j) \end{bmatrix} + Bu(i,j), \quad i,j \in \mathbb{Z}^+, \tag{1}$$

where $x^h(i,j) \in \mathbb{R}^{n_h}$ and $x^v(i,j) \in \mathbb{R}^{n_v}$ $n = n_h + n_v$ are the horizontal and the vertical state vectors, respectively; $u(i,j) \in \mathbb{R}^m$ is the control input. $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$ are given real matrices of appropriate dimensions and $E = diag\ E^h, E^v$ $\in \mathbb{R}^{n\times n}$ where $E^h \in \mathbb{R}^{n_h\times n_h}, E^v \in \mathbb{R}^{n_v\times n_v}$ and $rank(E^h) = r_h \leq n_h$, $rank(E^v) = r_v \leq n_v$ with $r = r_h + r_v < n$.

Initial condition of system (1) is specifed as

$$x^h(0,j) = x_0^h(j), 0 \leq j \leq T_1, \qquad x^v(i,0) = x_0^v(i), 0 \leq i \leq T_2, \tag{2a}$$

$$x^h(0,j) = 0, \forall j > T_1, \qquad x^v(i,0) = 0, \forall i > T_2, \tag{2b}$$

where $T_1, T_2 \in \mathbb{Z}^+$ are positive integers.

An SFC to stabilize system (1) will be designed in the form

$$u\ i,j\ = K\begin{bmatrix} x^h\ i,j \\ x^v\ i,j \end{bmatrix}. \tag{3}$$

Then, by incorporating the controller (3), the closed-loop system is obtained as

$$E\begin{bmatrix} x^h(i+1,j) \\ x^v(i,j+1) \end{bmatrix} = \ A + BK\ \begin{bmatrix} x^h(i,j) \\ x^v(i,j) \end{bmatrix}. \tag{4}$$

Let us introduce the following definitions.

**Definition 1** ([10]). The pair of matrice $(E, A)$ .is said to be regular if the two parameter polynomial $\det(E(z,s)-A)$ is not identically zero, and is causal if $\deg(\det(E(z,s)-A)) = rank(E)$, where $E(z,s) = diag(zE_h, sE_v)$.

**Definition 2** ([10]). The unforced system of (2) (i.e. $u = 0$) is said to be regular and causal if the pair $(E, A)$ is regular and causal.

**Definition 3** ([10]). The closed-loop system (4) is said to be internally stable if for any initial condition (2) it holds that

$$\lim_{q\to\infty} \sup\left\{\left\|\begin{matrix} x^h(i,j) \\ x^v(i,j) \end{matrix}\right\| : i + j = q\right\} = 0$$

**Definition 4** ([10]). The 2-D singular system (4) is said to be admissible if it is regular, causal and internally stable.

**Remak 1.** Since $rank(E) = r < n$. There exist nonsingular matrices $M, N$ such that

$$\hat{E} = MEN = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} \tag{5}$$

Let

$$\hat{A} = M(A + BK)N = \begin{bmatrix} \hat{A}_{11} & \hat{A}_{12} \\ \hat{A}_{21} & \hat{A}_{22} \end{bmatrix} \tag{6}$$

Using the decompositions (5) and (6), we obtain the following auxiliary result.

**Lemma 1 ([10]).** System (1) is regular and causal if the matrix $\hat{A}_{22}$ in the decomposition given in (6) is nonsingular.

**Lemma 2.** For any matrices $W_1, W_2$ of appropriate dimensions and a symmetric positive definite matrix $Q$ the following inequality holds

$$-W_1 Q W_1^T \le W_2^T W_1 + W_1^T W_2 + W_2^T Q^{-1} W_2$$

## 3. Main results

In this section, we first analyze the regularity, causality and stability of closed-loop system (4). Then, an SFC $u(i,j) = Kx(i,j)$ design is addressed.

**Theorem 1.** *The closed-loop 2-D singular system (4) is admissible if there exist symmetric positive definite matrix $P = diag(P^h, P^v)$ and a matrix $X$ of appropriate dimension by which the following LMI holds*

$$\begin{bmatrix} -E^T PE + X^T L^T A_c + A_c^T LX & A_c^T P \\ * & -P \end{bmatrix} < 0 \tag{7}$$

*where* $A_c = A + BK$ *and* $L = E^{T\perp}$ *is the null space matrix* $E^T$ *that is,* $E^T L = 0$ *and rank* $L = n - r$.

**Proof.** Firstly, we prove the closed-loop system (4) is regular and causal. Indeed, from (7), we have

$$-E^T PE + X^T L^T A_c + A_c^T LX < 0 \tag{8}$$

By pre- and post-multiplying both sides of (8) with $N^T$ and $N$, we obtain

$$\begin{aligned} &-N^T E^T M^T \left(M^{-T} PM^{-1}\right) MEN \\ &+N^T X^T L^T M^{-1} MA_c N + N^T A_c^T M^{-T} M^T LXN < 0 \end{aligned} \tag{9}$$

We decompose the following matrices

$$\begin{aligned} &\hat{P} = M^{-T} PM^{-1} = \begin{bmatrix} \hat{P}_{11} & \hat{P}_{12} \\ \hat{P}_{21} & \hat{P}_{22} \end{bmatrix}, \\ &\hat{X} = XN = \begin{bmatrix} \hat{X}_{11} & \hat{X}_{12} \end{bmatrix}, \\ &\hat{L} = M^{-T} L = \begin{bmatrix} \hat{L}_{11} \\ \hat{L}_{21} \end{bmatrix} \end{aligned} \tag{10}$$

It follows from $rank\ L = n - r$ and $E^T L = 0$ that

$$E^T L = N^T E^T M^T M^{-T} L = \hat{E}^T \hat{L} = 0,$$

which leads to $\hat{L}_{11} = 0$. Therefore, we can parameterize the matrix $L$ as $L = M^T \begin{bmatrix} 0 \\ L_{21} \end{bmatrix}$

Combining (5), (6) and (10), from (9), we have

$$-\hat{E}^T \hat{P}\hat{E} + \hat{X}^T \hat{L}^T \hat{A} + \hat{A}^T \hat{L}\hat{X} = \begin{bmatrix} \Pi_{11} & \Pi_{12} \\ * & \hat{X}_{12}^T \hat{L}_{21}^T \hat{A}_{22} + \hat{A}_{22}^T \hat{L}_{21} \hat{X}_{12} \end{bmatrix} < 0. \tag{11}$$

The LMI (11) also implies that

$$\hat{X}_{12}^T \hat{L}_{21}^T \hat{A}_{22} + \hat{A}_{22}^T \hat{L}_{21} \hat{X}_{12} < 0,$$

which ensures $\hat{A}_{22}$ is a nonsingular matrix. By Lemma 1, system (4) is regular and causal.

In the following, we will show that the closed-loop system (4) is internally stable. For this, we construct a 2-D Lyapunov function in the form

$$V(i,j) = \underbrace{x^{hT}(i,j)E^{hT}P^{h}E^{h}x^{h}(i,j)}_{V^{h}(i,j)} + \underbrace{x^{vT}(i,j)E^{vT}P^{v}E^{v}x^{v}(i,j)}_{V^{v}(i,j)}. \tag{12}$$

First, the differences of $V^{h}(i,j)$, $V^{v}(i,j)$ along trajectories of system (4) is given by

$$V^{h}(i+1,j) - V^{h}(i,j) = x^{hT}(i+1,j)E^{hT}P^{h}E^{h}x^{h}(i+1,j) - x^{hT}(i,j)E^{hT}P^{h}E^{h}x^{h}(i,j), \tag{13}$$

$$V^{v}(i,j+1) - V^{v}(i,j) = x^{vT}(i,j+1)E^{vT}P^{v}E^{v}x^{v}(i,j+1) - x^{vT}(i,j)E^{vT}P^{v}E^{v}x^{v}(i,j). \tag{14}$$

For the brevity, we denote the following augmented vectors

$x(i,j) = \begin{bmatrix} x^{h}(i,j) \\ x^{v}(i,j) \end{bmatrix}, \quad x_{+}(i,j) = \begin{bmatrix} x^{h}(i+1,j) \\ x^{v}(i,j+1) \end{bmatrix}.$ Then, from (4), we have

$$\begin{aligned} & x^{hT}(i+1,j)E^{hT}P^{h}E^{h}x^{h}(i+1,j) + x^{vT}(i,j+1)E^{vT}P^{v}E^{v}x^{v}(i,j+1) \\ & = x_{+}^{T}(i,j)E^{T}PEx_{+}(i,j) \\ & = x^{T}(i,j)A_{c}^{T}PA_{c}x(i,j) \end{aligned} \tag{15}$$

and

$$\begin{aligned} & x^{hT}(i,j)E^{hT}P^{h}E^{h}x^{h}(i,j) + x^{vT}(i,j)E^{vT}P^{v}E^{v}x^{v}(i,j) \\ & = x^{T}(i,j)E^{T}PEx(i,j). \end{aligned} \tag{16}$$

On the other hand, since $rank(L) = n - r$ and $E^{T}L = 0$, for any matrices $X$ of appropriate dimension, the following zero-equation holds

$$2x(i,j)X^{T}L^{T}Ex_{+}(i,j) = 2x(i,j)X^{T}L^{T}(A+BK)x(i,j) = 0. \tag{17}$$

From (15) to (17) we obtain

$$\begin{aligned} & V^{h}(i+1,j) + V^{v}(i,j+1) - \left(V^{h}(i,j) + V^{v}(i,j)\right) \\ & = x^{T}(i,j)\left(-E^{T}PE + X^{T}L^{T}A_{c} + A_{c}^{T}LX + A_{c}^{T}PA_{c}\right)x(i,j). \end{aligned} \tag{18}$$

By the Schur complement lemma supply a reference for this lemma, it follows from (7) that

$$-E^T PE + X^T L^T A_c + A_c^T LX + A_c^T PA_c < 0.$$

Thus, there exists a positive number $\lambda_0$ such that

$$V^h(i+1, j) + V^v(i, j+1) - \left(V^h(i, j) + V^v(i, j)\right) \le -\lambda_0 \|x(i, j)\|^2, \quad \forall i, j \in \mathbb{Z}^+. \tag{19}$$

For any positive integer $q$, let $E(q) = \sum_{(i,j)\in\Gamma(q)} V(i, j)$ denote the energy of the functional $V(i, j)$ given in (12) stored along the diagonal line $\Gamma(q) = \{(i, j) : i + j = q, i \ge 0, j \ge 0\}$. It can be deduced from (19) that

$$\begin{aligned}
\sum_{(i,j)\in\Gamma(q+1)} V(i, j) &= \sum_{(i,j)\in\Gamma(q+1)} V^h(i, j) + V^v(i, j) \\
&= V^h(1, q) + V^h(2, q) + \cdots + V^h(q+1, 1) \\
&\quad + V^v(q, 1) + V^v(q-1, 2) + \cdots + V^v(0, q+1) \\
&\le V^h(0, q) + V^h(1, q) + \cdots + V^h(q, 1) \\
&\quad + V^v(q, 0) + V^v(q-1, 0) + \cdots + V^v(0, q) - \lambda_0 \sum_{(i,j)\in\Gamma(q)} \|x(i, j)\|^2 \\
&= \sum_{(i,j)\in\Gamma(q)} V^h(i, j) + V^v(i, j) - \lambda_0 \sum_{(i,j)\in\Gamma(q)} \|x(i, j)\|^2
\end{aligned}$$

Therefore,

$$E(q+1) \le E(q) - \lambda_0 \sum_{(i,j)\in\Gamma(q)} \|x(i, j)\|^2 . \tag{20}$$

It can be verified from (20) that $E(q)$ is a nonnegative decreasing sequence.

This shows that there exists finite limit $E(\infty) = \lim_{q\to\infty} E(q)$. In addition to this

$$\lambda_0 \sum_{(i,j)\in\Gamma(q)} \|x(i, j)\|^2 \le E(q) - E(q+1) \to E(\infty) - E(\infty) = 0$$

as $q \to \infty$, which shows that system (4) is internally stable. The proof is completed.

The stabilization conditions of system (1) are presented in the following theorem.

**Theorem 2.** *The closed-loop system (4)} is admissible if there exist a symmetric positive definite matrix* $\mathcal{P} = diag(\mathcal{P}^h, \mathcal{P}^v)$*, an invertible matrix* $\mathcal{X}$ *and any matrix* $\mathcal{W}$ *of appropriate dimension, such that the following LMI holds*

$$\begin{bmatrix} \mathcal{P} + Sym\left\{E\mathcal{X} + L^T A\mathcal{X} + L^T B\mathcal{W}\right\} & \mathcal{X}^T A^T + \mathcal{W}^T B^T \\ * & -\mathcal{P} \end{bmatrix} < 0, \tag{21}$$

*where*

$L = \left(E^T\right)^{\perp}$ *and* $Sym\{.\}$ *denotes the symmetric operator, that is,* $Sym\{M\} = M + M^T = T$. *A desired SFC gain is obtained as* $K = \mathcal{W}\mathcal{X}^{-1}$.

**Proof.** According to Theorem 1, we choose $X = \mathrm{diag}(X^h, X^v)$ as an invertible matrix. By pre- and post-multiplying both sides of (7) with $diag\{X^{-T}, P^{-1}\}$ and its transpose, respectively, we obtain

$$\begin{bmatrix} -X^{-T}E^T PEX^{-1} + Sym\left\{L^T(A+BK)X^{-1}\right\} & X^{-T}(A+BK)^T \\ * & -P^{-1} \end{bmatrix} < 0. \quad (22)$$

In addition, by utilizing the matrix inequality given in Lemma 2, we have

$$-X^{-T}E^T PEX^{-1} \le X^{-T}E^T + EX^{-1} + P^{-1}. \quad (23)$$

Now, we let $\mathcal{X} = X^{-1}, \mathcal{P} = P^{-1}$ and $\mathcal{W} = KX^{-1}$. Combining (23) to (22), we get the LMI condition (21). In addition, the controller gain can be obtained as

$$K = \mathcal{W}\mathcal{X}^{-1}.$$

The proof is completed.

## 4. Numerical example

**Example 1:**

Consider system (1) with the following parameters:

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, A = \begin{bmatrix} -0.14 & 0.5 & 1.12 \\ 1.2 & 0.13 & 0.01 \\ -0.02 & -0.15 & -0.15 \end{bmatrix}, \quad B = \begin{bmatrix} 0.01 \\ 0.01 \\ -0.02 \end{bmatrix}.$$

It is easy to verify that $\hat{A}_{22} = \begin{bmatrix} -0.2 & -0.48 \\ 0.0125 & 0.22 \end{bmatrix}$ is a nonsingular matrix, which proves that system (1) is regular and causal.

According to Theorem 2 and by using the Matlab LMI Toolbox, we find the matrices follows

$$P = \begin{bmatrix} 49.2983 & 0 & 0 \\ 0 & 128.4781 & 2.1011 \\ 0 & 2.1011 & 88.6602 \end{bmatrix}, X = \begin{bmatrix} -61.3825 & 3.3493 & -42.5257 \\ 0.3009 & -109.9202 & 223.4964 \\ -11.0355 & 45.0401 & -118.7774 \end{bmatrix}.$$

$$W = \begin{bmatrix} 165.4 & 484.9 & 1511.3 \end{bmatrix}$$

and the following controller gain can be obtained

$$K = \begin{bmatrix} 17.7516 & -51.0423 & -115.1230 \end{bmatrix}.$$

Which shows that the 2-D singular system is admissible.

## 5. Conclusion

This paper has dealt with the stabilization problem via state-feedback control of 2-D singular Roesser systems. Sufficient stability conditions in terms of LMIs have been derived based on a 2-D Lyapunov function scheme and utilizing zero-type free matrix equations. On the basis of the analysis result, a stabilizing SFC can be implemented using derived tractable LMIs conditions.

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